Maximal Ratio Combining with Correlated Rayleigh Fading Channels

"They say we're not placing enough emphasis on diversity."

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Abstract—This project examines maximal ratio combining (MRC) in independent and correlated Rayleigh fading channels. While channel correlation is a key factor in the performance of linear combining, significant performance improvement can still be achieved with semi-correlated fading channels with $\rho \approx .6$. Also, the antenna separation required at the base station (BS) and mobile station (MS) to achieve the same degree of signal decorrelation (and therefore performance improvement) is not the same due to the inherently different environments around the BS and MS.

OUTLINE

1. INTRODUCTION

In mobile wireless communications, multipath fading can cause constructive and destructive interference. A popular method to mitigate the effects of multipath fading is diversity, a process of obtaining multiple independent signal branches through many dimensions including space, polarization, frequency, and time [1]. The collection of independently fading signal branches can then be combined in a variety of ways to improve the received signal-to-noise ratio (SNR).

The three most prevalent space diversity combining techniques are selection combining (SC), equal gain combining (EGC), and MRC. MRC co-phases the signal branches, weights them according to their respective SNRs, and then takes their sum. MRC is the most complex combining technique, but also yields the highest SNR [4]. This project will analyze the performance of MRC in both independent and correlated fading channels.

For space diversity combining, one key assumption is that the antennas are spaced far apart enough such that the received signal branch at each antenna will experience independent channel fading [2]. With the decreasing size of mobile devices, the space required for antenna decorrelation is not met [3]. In this project we examine through analysis and simulation how much antenna separation is required for satisfactory fading decorrelation. Furthermore, due to the dissimilarity between the base station (BS) and mobile station (MS) surroundings, the amount of antenna separation required at the BS and MS for a given correlation coefficient will also be examined.

2. MRC WITH INDEPENDENT FADING

2.1. Signal-To-Noise Ratio

We will consider the case of MRC in a single-input multiple-output (SIMO) system with M receive antennas. In order to examine the impact of MRC, we will derive an expression for the SNR with MRC, γ_Σ, in terms of M as well as the SNR of a single-input single-output (SISO) antenna configuration (or equivalently, the SNR of a single branch of the MRC system, γ_m). This will show whether an MRC system can satisfy a target BER constraint at a higher or lower SNR than a system without MRC.

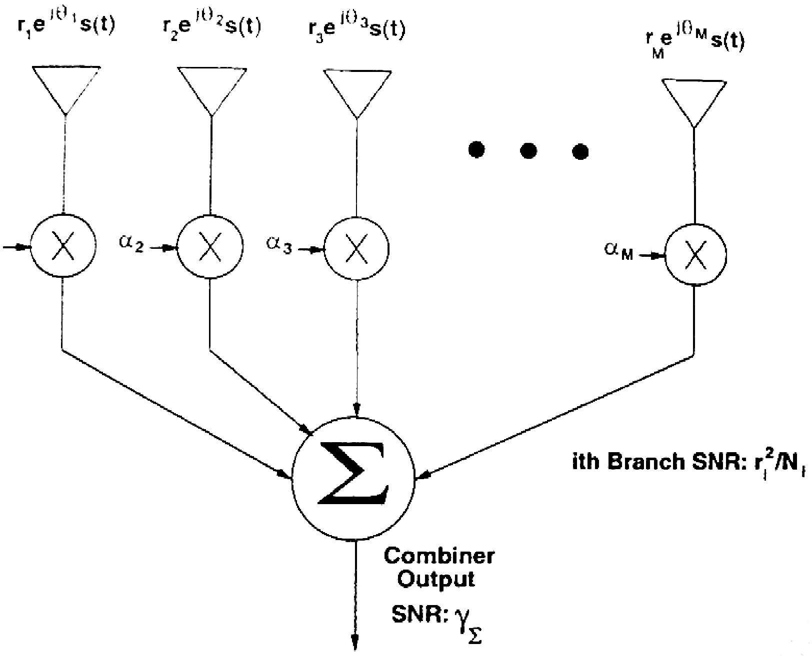

Figure 1: Linear Combiner [2]

Figure 1 shows a diagram of a linear combiner that implements either selection combining (SC), equal gain combining, (EGC), or MRC [2]. We will focus on MRC. Given M receiver antennas and ignoring noise, the received signal on the mth antenna can be expressed as $r_m e^{j\theta_m} s(t)$, where r_m is the envelope magnitude, θ_m is the phase of the received signal branch, and $s(t)$ is the transmitted signal. The signal branch at each antenna is then multiplied by a complex number $\alpha_m = a_m e^{-j\theta_m}$ in the linear combiner such that each of the signal branches are cophased (i.e. all branches have zero phase). Then the cophased branches are summed, and the resultant signal envelope is $r_\Sigma = \sum_{m=1}^{M} a_m r_m$.

The combiner output signal power is then:

$$r_\Sigma^2 = \left(\sum_{m=1}^{M} a_m r_m\right)^2.$$

To compute the SNR, we also need the total noise power. Assuming that each antenna has the same noise power *N* (which is plausible due to the similar construction of each receiver), and after the noise in each branch is multiplied by its respective α_m, the combiner output noise power is:

$$N_\Sigma = \sum_{m=1}^{M} a_m^2 N.$$

The combiner output SNR, γ_Σ, can now be expressed as:

$$\gamma_\Sigma = \frac{r_\Sigma^2}{N_\Sigma} = \frac{\left(\sum_{m=1}^{M} a_m r_m\right)^2}{\sum_{m=1}^{M} a_m^2 N} = \frac{1}{N}\cdot\frac{\left(\sum_{m=1}^{M} a_m r_m\right)^2}{\sum_{m=1}^{M} a_m^2}.$$

As MRC system designers we want to maximize γ_Σ. After cophasing, the only variables left for designers to work with are the a_m s, which can be found by invoking the Swartz inequality [6], which gives us [7]:

$$\left|\sum_{m=1}^{M} a_m r_m\right|^2 \le \left(\sum_{m=1}^{M} |r_m|^2\right)\left(\sum_{m=1}^{M} |a_m|^2\right)$$

Assuming all a_m s and r_m s are positive, the Swartz inequality becomes:

$$\left(\sum_{m=1}^{M} a_m r_m\right)^2 \le \left(\sum_{m=1}^{M} r_m^2\right)\left(\sum_{m=1}^{M} a_m^2\right)$$

Substituting into the equation for γ_Σ, we get:

$$\gamma_\Sigma = \frac{1}{N}\cdot\frac{\left(\sum_{m=1}^{M} a_m r_m\right)^2}{\sum_{m=1}^{M} a_m^2} \le \frac{1}{N}\cdot\frac{\left(\sum_{m=1}^{M} r_m^2\right)\left(\sum_{m=1}^{M} a_m^2\right)}{\sum_{m=1}^{M} a_m^2} = \frac{1}{N}\cdot\left(\sum_{m=1}^{M} r_m^2\right).$$

The above relationship shows that the maximum γ_Σ can be obtained if we select values for the a_m s such that $\gamma_\Sigma = \frac{1}{N}\cdot\frac{\left(\sum_{m=1}^{M} a_m r_m\right)^2}{\sum_{m=1}^{M} a_m^2} = \frac{1}{N}\cdot\left(\sum_{m=1}^{M} r_m^2\right)$. By inspection we see that if we set $a_m = Kr_m$ for some constant *K*, then we will maximize γ_Σ, obtaining:

$$\gamma_\Sigma = \frac{1}{N}\cdot\frac{\left(\sum_{m=1}^{M} K r_m^2\right)^2}{\sum_{m=1}^{M}(K r_m)^2} = \frac{1}{N}\cdot\frac{K^2\left(\sum_{m=1}^{M} r_m^2\right)^2}{K^2\sum_{m=1}^{M} r_m^2} = \frac{1}{N}\cdot\left(\sum_{m=1}^{M} r_m^2\right) = \sum_{m=1}^{M}\frac{r_m^2}{N} = \sum_{m=1}^{M}\gamma_m\,.$$

But, $\frac{r_m^2}{N} = \gamma_m$ is the SNR of each received signal branch, so in an MRC system γ_Σ is just equal to the sum of all the γ_m s:

$$\gamma_\Sigma = \sum_{m=1}^{M}\gamma_m$$

Furthermore, if all branches have equal average SNR $\overline{\gamma}_m$, then:

$$\boxed{\overline{\gamma}_\Sigma = M\cdot\overline{\gamma}_m}$$

A key point from this result is that a minimum SNR requirement can be met with MRC, even if none of the individual branch SNRs meet the minimum requirement.

2.2. Probability of Bit Error

First we express the probability of error (i.e. BER) conditioning upon $\{\gamma_m\}$. This is given by the BER expression for BPSK in AWGN [2]:

$$\begin{aligned} P_b\left(\text{error}\,|\,\{\gamma_m\}\right) &= \mathbf{Q}\left(\sqrt{2\gamma_\Sigma}\right) = \frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{\gamma_\Sigma}{\sin^2\phi}\right)d\phi \\ &= \frac{1}{\pi}\int_0^{\pi/2}\prod_{m=1}^{M}\exp\left(-\frac{\gamma_m}{\sin^2\phi}\right)d\phi,\ \gamma_\Sigma \ge 0 \end{aligned}$$

where we have utilized Craig's alternate representation of the Q-function. From here, the unconditional BER can be found by averaging the conditional BER over the joint probability density function (PDF) of the branch SNRs $\{\gamma_m\}$. Assuming independent fading on each branch, the unconditional BER consists of M integrals of the following form:

$$P_b\left(\text{error}\right) = \underbrace{\int_0^\infty\int_0^\infty\cdots\int_0^\infty}_{M\text{-fold}} P_b\left(\text{error}|\{\gamma_\mathrm{m}\}\right)\prod_{m=1}^{M} p_{\gamma_m}\left(\gamma_m;\overline{\gamma}_m\right)d\gamma_1 d\gamma_2\cdots d\gamma_M$$

where the term after the semicolon is the parameter for the Rayleigh distribution. Substituting in for the conditional probability of error gives:

$$P_b(\text{error}) = \underbrace{\int_0^\infty\int_0^\infty\cdots\int_0^\infty}_{M\text{-fold}} \frac{1}{\pi}\int_0^{\pi/2}\prod_{m=1}^{M}\exp\left(-\frac{\gamma_m}{\sin^2\phi}\right)d\phi\prod_{m=1}^{M}p_{\gamma_m}\left(\gamma_m;\overline{\gamma}_m\right)d\gamma_1 d\gamma_2\cdots d\gamma_M$$

$$= \underbrace{\int_0^\infty\int_0^\infty\cdots\int_0^\infty}_{M\text{-fold}} \frac{1}{\pi}\int_0^{\pi/2}\prod_{m=1}^{M}\exp\left(-\frac{\gamma_m}{\sin^2\phi}\right)p_{\gamma_m}\left(\gamma_m;\overline{\gamma}_m\right)d\phi d\gamma_1 d\gamma_2\cdots d\gamma_M$$

Because the integrand is absolutely integrable, we can interchange the order of integration in our expression for $P_b(\text{error})$ [5]. Also, assuming identically Rayleigh-distributed channels with the same average SNR on each branch, i.e. $p_{\gamma_m}\left(\gamma_m;\overline{\gamma}_m\right) = p_\gamma\left(\gamma;\overline{\gamma}\right)$ and $\overline{\gamma}_m = \overline{\gamma}$ for $\forall\ m$, we get:

$$P_b(\text{error}) = \int_0^{\pi/2} \underbrace{\int_0^\infty\int_0^\infty\cdots\int_0^\infty}_{M\text{-fold}} \frac{1}{\pi}\prod_{m=1}^{M}\exp\left(-\frac{\gamma_m}{\sin^2\phi}\right)p_{\gamma_m}\left(\gamma_m;\overline{\gamma}_m\right)d\gamma_1 d\gamma_2\cdots d\gamma_M d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi/2}\left(\mathrm{M}_r\left(-\frac{1}{\sin_2\phi};\overline{\gamma}\right)\right)^M d\phi$$

where $\mathrm{M}_r\left(-\frac{1}{\sin_2\phi};\overline{\gamma}\right)$ is the moment generating function (MGF) of the Rayleigh distribution of the SNR of each branch, $p_\gamma\left(\gamma;\overline{\gamma}\right)$, assuming BPSK modulation. The expression for the MGF is:

$$\mathrm{M}_r\left(-\frac{1}{\sin^2\phi};\overline{\gamma}\right) = \left(1+\frac{\overline{\gamma}}{\sin^2\phi}\right)^{-1}$$

Substituting in the MGF, we arrive at our final expression for the BER of a SIMO MRC system with M receive antennas:

$$\boxed{P_{b\ (\text{MRC, indep. fading})}(\text{error}) = \frac{1}{\pi}\int_0^{\pi/2}\left(1+\frac{\overline{\gamma}}{\sin^2\phi}\right)^{-M} d\phi}.$$

2.3. Analytical BER Results

MATLAB was used to calculate the following analytical BER results:

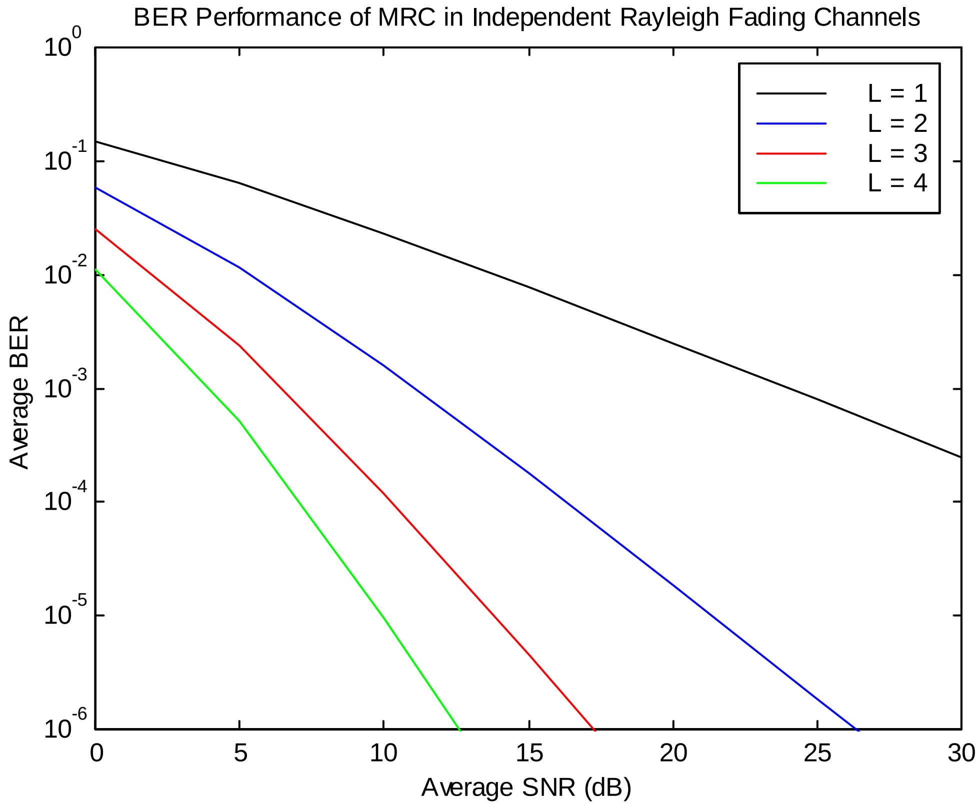

Figure 2: Analytical BER versus SNR for 1, 2, 3, and 4 receive antennas

Note that as the number of receive antennas increases, the slope of the BER curve gets steeper, which is consistent with our SNR analysis.

2.4. Simulation BER Results

The BPSK modulated transmitted signal $x = +1$ or -1 is multiplied by a Rayleigh fading channel and then AWGN is added to the result, forming the received signal. The matrix equation for the received signal branches is shown below:

$$\mathbf{y} = \mathbf{Hx} + \mathbf{N}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} x + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is the received signal vector, $\mathbf{H} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ is the channel matrix, and $\mathbf{N} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ is the noise matrix for two receive antennas. The MRC receiver then cophases and weights the received signal branches by multiplying it by the conjugate transpose of the channel matrix (assuming the receiver knows the channel perfectly) and normalizing the result:

$$\mathbf{H}^{*}\mathbf{y} = \left(|h_1|^2 + |h_2|^2\right)x + \mathbf{H}^{*}\mathbf{N}$$

$$\hat{x} = \frac{\mathbf{H}^{*}\mathbf{y}}{|h_1|^2 + |h_2|^2} = x + \frac{\mathbf{H}^{*}\mathbf{N}}{|h_1|^2 + |h_2|^2}.$$

$\hat{x}$ is then decoded into a +1 or –1 using a maximum likelihood decoder.

Here are the simulation BER results from my previous work [8].

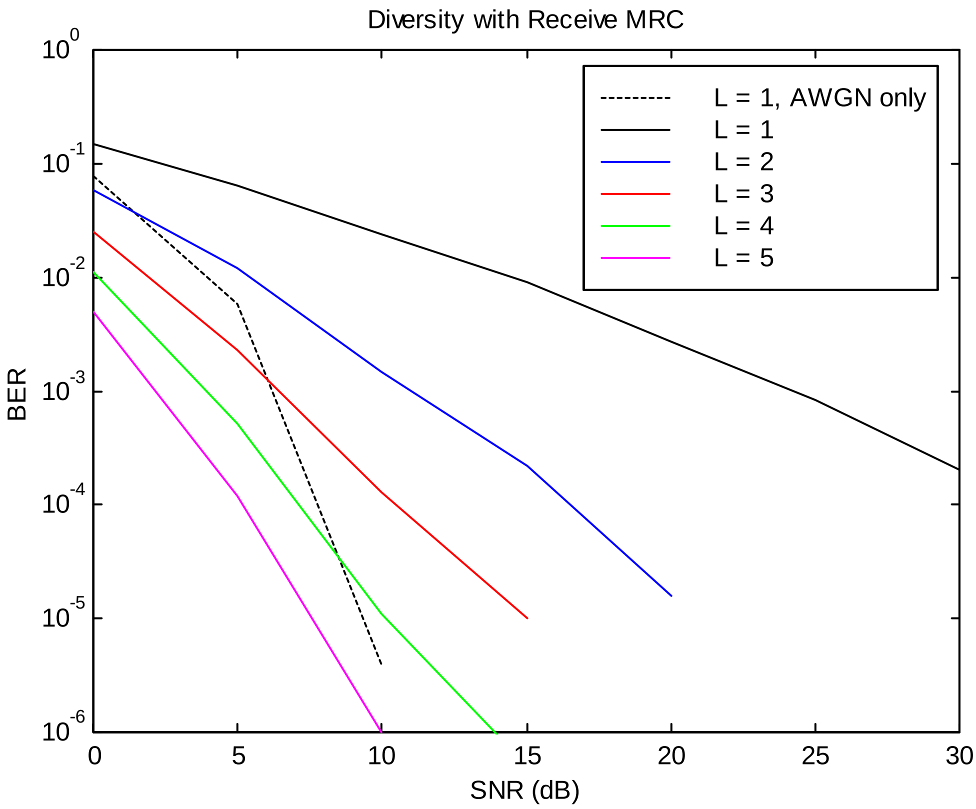

Figure 3: Simulation BER versus SNR for 1, 2, 3, 4, and 5 receive antennas (includes BER in an AWGN channel without fading)

2.5. Comparison of Simulation to Analytical Results

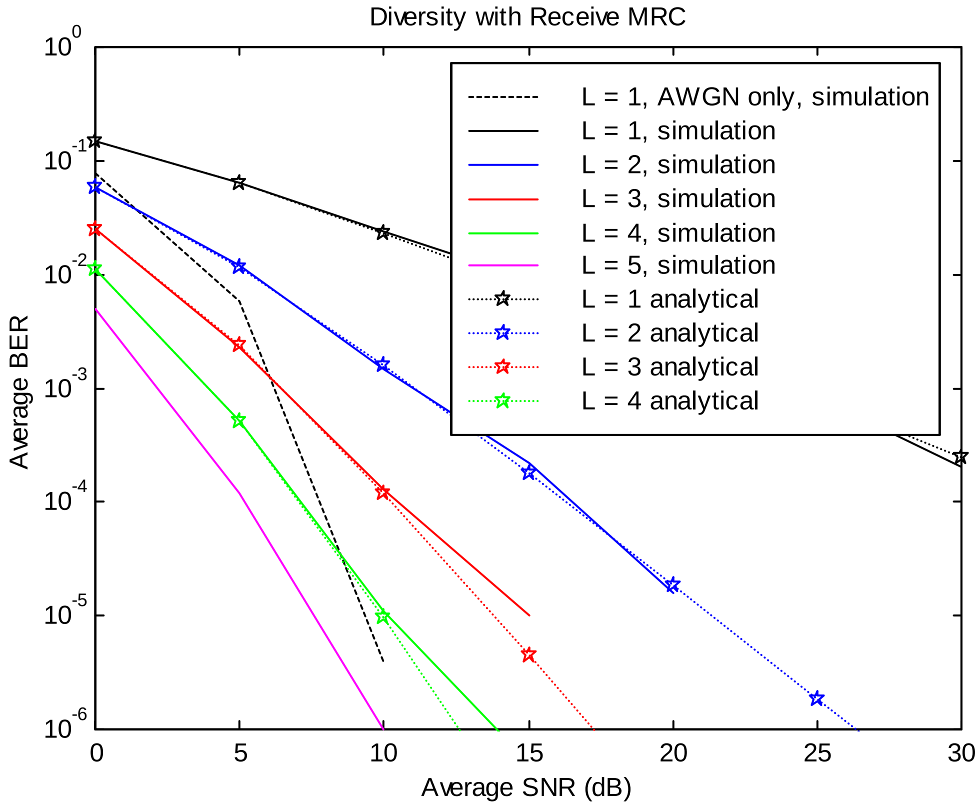

Figure 4: Comparison of Simulation to Analytical BER Results

From this plot we see that the simulation produces BER results very close to those predicted by our analysis. The slight differences can be attributed to an insufficiently large number of trials to average over (which was limited due to extremely long simulation run-times).

3. MRC WITH CORRELATED FADING

3.1. Signal-to-Noise Ratio

Consider an MRC system with 2 receive antenna branches with correlated Rayleigh fading channels. Here we discuss a method described by [3] to transform two correlated signal branches with respective SNRs γ_1 and γ_2 into two independent signals with respective SNRs γ_3 and γ_4. As before, the received signal at antenna branch ℓ ignoring noise is expressed as $r_\ell e^{j\theta_\ell} s(t)$ where r_ℓ is a Rayleigh-distributed random variable. If we include noise into our signal model, we can express the received signals at antenna 1 and antenna 2 respectively as:

$$y_1(t) = r_1 e^{j\theta_1} s(t) + n_1(t)$$

$$y_2(t) = r_2 e^{j\theta_1} s(t) + n_2(t)$$

where $n_\ell(t)$ is AWGN with zero mean for $\ell = 1,2$. The fading envelope $r_\ell e^{j\theta_\ell}$ can be expressed as a sum of the in-phase and quadrature components of the envelope:

$$r_\ell e^{j\theta_\ell} = r_{\ell,I} + j r_{\ell,Q}$$

where $r_{\ell,I}$ and $r_{\ell,Q}$ are independent zero mean Gaussian random variables with variance σ_G^2. Also, by definition, the correlation $\rho_{I,Q}$ between the in-phase and quadrature component for each fading envelope is always zero. We can now express the received signals as:

$$y_1(t) = \left(r_{1,I} + j r_{1,Q}\right) s(t) + n_1(t)$$

$$y_2(t) = \left(r_{2,I} + j r_{2,Q}\right) s(t) + n_2(t)$$

This shows that in order to examine the correlation between the fading envelopes $r_1 e^{j\theta_1}$ and $r_2 e^{j\theta_2}$, we can equivalently examine the correlation ρ_I between the in-phase components $r_{1,I}$ and $r_{2,I}$ as well as the correlation ρ_Q between the quadrature components $r_{1,Q}$ and $r_{2,Q}$. For this analysis we assume $\rho_I = \rho_Q = \rho_G$ where *G* denotes the correlation between two Gaussian distributed fading envelope components. For $\rho_G = 0$, we have uncorrelated fading channels described by the covariances between each combination of two Gaussian random variables [14]:

$$\begin{aligned} \mathrm{Cov}\left(r_{1,I}, r_{2,I}\right) &= \rho\left(r_{1,I}, r_{2,I}\right) \cdot \sigma_{r_{1,I}} \sigma_{r_{2,I}} \\ &= \rho_I \sigma_G^2 = \rho_G \sigma_G^2 = 0, \\ \mathrm{Cov}\left(r_{1,Q}, r_{2,Q}\right) &= 0. \end{aligned}$$

Similarly, $\rho_Q = \rho_{I,Q} = 0$, which gives us:

$$\mathrm{Cov}\left(r_{1,I}, r_{1,Q}\right) = \mathrm{Cov}\left(r_{1,I}, r_{2,Q}\right) = \mathrm{Cov}\left(r_{1,I}, r_{2,Q}\right) = \mathrm{Cov}\left(r_{2,I}, r_{1,Q}\right) = 0.$$

For $0 < \rho_G \le 1$, all cross-correlations remain equal to zero except $\mathrm{Cov}\left(r_{1,I}, r_{2,I}\right)$ and $\mathrm{Cov}\left(r_{1,Q}, r_{2,Q}\right)$:

$$\mathrm{Cov}\left(r_{1,I}, r_{2,I}\right) = \mathrm{Cov}\left(r_{1,Q}, r_{2,Q}\right) = \rho_G \sigma_G^2.$$

Assuming that the average SNR is the same at both receive antennas, we have:

$$\overline{\gamma}_\ell = \frac{E\left[r_\ell^2\right]}{E\left[n_\ell^2(t)\right]} = \frac{\mathrm{var}\left[r_\ell\right] + E^2\left[r_\ell\right]}{N_\ell} = \frac{2\sigma_G^2}{N} = \overline{\gamma} \text{ for } \ell = 1,2.$$

We now introduce a transformation from the correlated received signal branches $y_1(t)$ and $y_2(t)$ into independent signal branches $y_3(t)$ and $y_4(t)$. Define the following transformation matrix *T*:

$$T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

If we apply the *T* to the $y_1(t)$ and $y_2(t)$, we will get $y_3(t)$ and $y_4(t)$ expressed as linear combinations of $y_1(t)$ and $y_2(t)$:

$$\begin{aligned}
\begin{bmatrix} y_3(t) \\ y_4(t) \end{bmatrix} = T\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} y_1(t) + \frac{\sqrt{2}}{2} y_2(t) \\ -\frac{\sqrt{2}}{2} y_1(t) + \frac{\sqrt{2}}{2} y_2(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2}\left[\left(r_{1,I} + jr_{1,Q}\right)s(t) + n_1(t)\right] + \frac{\sqrt{2}}{2}\left[\left(r_{2,I} + jr_{2,Q}\right)s(t) + n_2(t)\right] \\ -\frac{\sqrt{2}}{2}\left[\left(r_{1,I} + jr_{1,Q}\right)s(t) + n_1(t)\right] + \frac{\sqrt{2}}{2}\left[\left(r_{2,I} + jr_{2,Q}\right)s(t) + n_2(t)\right] \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} r_{1,I}s(t) + j\frac{\sqrt{2}}{2} r_{1,Q}s(t) + \frac{\sqrt{2}}{2} n_1(t) + \frac{\sqrt{2}}{2} r_{2,I}s(t) + j\frac{\sqrt{2}}{2} r_{2,Q}s(t) + \frac{\sqrt{2}}{2} n_2(t) \\ -\frac{\sqrt{2}}{2} r_{1,I}s(t) - j\frac{\sqrt{2}}{2} r_{1,Q}s(t) - \frac{\sqrt{2}}{2} n_1(t) + \frac{\sqrt{2}}{2} r_{2,I}s(t) + j\frac{\sqrt{2}}{2} r_{2,Q}s(t) + \frac{\sqrt{2}}{2} n_2(t) \end{bmatrix} \\
&= \begin{bmatrix} \left[\left(\frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}\right) + j\left(\frac{\sqrt{2}}{2} r_{1,Q} + \frac{\sqrt{2}}{2} r_{2,Q}\right)\right]s(t) + \frac{\sqrt{2}}{2} n_1(t) + \frac{\sqrt{2}}{2} n_2(t) \\ \left[\left(-\frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}\right) + j\left(-\frac{\sqrt{2}}{2} r_{1,Q} + \frac{\sqrt{2}}{2} r_{2,Q}\right)\right]s(t) - \frac{\sqrt{2}}{2} n_1(t) + \frac{\sqrt{2}}{2} n_2(t) \end{bmatrix} \\
&= \begin{bmatrix} \left(r_{3,I} + jr_{3,Q}\right)s(t) + n_3(t) \\ \left(r_{4,I} + jr_{4,Q}\right)s(t) + n_4(t) \end{bmatrix}
\end{aligned}$$

where

$$r_{3,I} = \frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}$$

$$r_{3,Q} = \frac{\sqrt{2}}{2} r_{1,Q} + \frac{\sqrt{2}}{2} r_{2,Q}$$

$$n_3(t) = \frac{\sqrt{2}}{2} n_1(t) + \frac{\sqrt{2}}{2} n_2(t)$$

$$r_{4,I} = -\frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}$$
$$r_{4,Q} = -\frac{\sqrt{2}}{2} r_{1,Q} + \frac{\sqrt{2}}{2} r_{2,Q}$$
$$n_4(t) = -\frac{\sqrt{2}}{2} n_1(t) + \frac{\sqrt{2}}{2} n_2(t)$$

All of the above expressions are sums of zero mean Gaussian random variables, which means that the sums are also zero mean Gaussian random variables. Furthermore:

$$\begin{aligned}
\mathrm{Cov}\left(r_{3,I}, r_{4,I}\right) &= E\left[r_{3,I} r_{4,I}\right] - E\left[r_{3,I}\right] E\left[r_{4,I}\right] \\
&= E\left[\left(\frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}\right)\left(-\frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}\right)\right] - \\
&\quad E\left[\left(\frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}\right)\right] E\left[\left(-\frac{\sqrt{2}}{2} r_{1,I} + \frac{\sqrt{2}}{2} r_{2,I}\right)\right] \\
&= E\left[-\frac{1}{2} r_{1,I}^2 + \frac{1}{2} r_{2,I}^2\right] - \\
&\quad \left(\frac{\sqrt{2}}{2} E\left[r_{1,I}\right] + \frac{\sqrt{2}}{2} E\left[r_{2,I}\right]\right)\left(-\frac{\sqrt{2}}{2} E\left[r_{1,I}\right] + \frac{\sqrt{2}}{2} E\left[r_{2,I}\right]\right) \\
&= -\frac{1}{2} E\left[r_{1,I}^2\right] + \frac{1}{2} E\left[r_{2,I}^2\right] - \left(-\frac{1}{2} E^2\left[r_{1,I}\right] + \frac{1}{2} E^2\left[r_{2,I}\right]\right) \\
&= -\frac{1}{2} E\left[r_{1,I}^2\right] + \frac{1}{2} E\left[r_{2,I}^2\right] + \frac{1}{2} E^2\left[r_{1,I}\right] - \frac{1}{2} E^2\left[r_{2,I}\right] \\
&= \frac{1}{2}\left(\sigma_G^2 - \sigma_G^2\right) = 0.
\end{aligned}$$

Since $r_{3,I}$ and $r_{4,I}$ are Gaussian and uncorrelated, this means that they are independent. Similarly, $r_{3,Q}$, $r_{4,Q}$, $n_3(t)$, and $n_4(t)$ are mutually independent. The variances of $r_{3,I}$, $r_{4,I}$, $r_{3,Q}$, and $r_{4,Q}$ are [9]:

$$\begin{aligned}
\mathrm{var}\left[r_{3,I}\right] &= \mathrm{var}\left[r_{3,Q}\right] \\
&= \sigma_3^2 = \frac{1}{2}\sigma_G^2 + 2\rho_G \cdot \frac{1}{2}\sigma_G^2 + \frac{1}{2}\sigma_G^2 = \sigma_G^2 + \rho_G \sigma_G^2 = \left(1+\rho_G\right)\sigma_G^2 \\
\mathrm{var}\left[r_{4,I}\right] &= \mathrm{var}\left[r_{4,Q}\right] \\
&= \sigma_4^2 = \frac{1}{2}\sigma_G^2 - 2\rho_G \cdot \frac{1}{2}\sigma_G^2 + \frac{1}{2}\sigma_G^2 = \sigma_G^2 - \rho_G \sigma_G^2 = \left(1-\rho_G\right)\sigma_G^2.
\end{aligned}$$

The noise powers are:

$$N_3 = \frac{1}{2} N_1 + \frac{1}{2} N_2 = N$$
$$N_4 = \frac{1}{2} N_1 + \frac{1}{2} N_2 = N.$$

So the average SNRs of $y_3(t)$ and $y_4(t)$ are:

$$\boxed{\begin{aligned}\bar{\gamma}_3 &= \frac{2(1+\rho_G)\sigma_G^2}{N} = (1+\rho_G)\bar{\gamma} \\ \bar{\gamma}_4 &= \frac{2(1-\rho_G)\sigma_G^2}{N} = (1-\rho_G)\bar{\gamma}\end{aligned}}$$

This shows that two correlated fading signal branches with equal average SNR can be transformed into two independent fading signal branches with unequal average SNRs dependent on the fading correlation coefficient. Clearly this will lead to performance degradation if the MRC continues to operate as if it were receiving independently fading signals.

1.2. Probability of Bit Error

First we define the moment generating function (MGF) of γ_Σ [2]:

$$C_{\bar{\gamma}_\Sigma}(s) = \int_0^\infty e^{s\gamma_\Sigma} p(\gamma_\Sigma)\, d\gamma_\Sigma$$

The average BER is given by:

$$\overline{P}_b = \int_0^\infty P_b(\gamma_\Sigma) p(\gamma_\Sigma)\, d\gamma_\Sigma$$

The expression for $P_b(\gamma_\Sigma)$ for BPSK can be simplified to [10]:

$$P_b(\gamma_\Sigma) = \mathbf{Q}\left(\sqrt{2\gamma_\Sigma}\right) = \frac{1}{2\pi}\int_1^\infty \frac{1}{s\sqrt{s-a}} \exp\left[-\gamma_\Sigma s\right] ds\, .$$

Utilizing the MGF of γ_Σ and interchanging the order of integration, we get following expression for the average BER:

$$\begin{aligned}\overline{P}_b &= \int_0^\infty \frac{1}{2\pi}\int_1^\infty \frac{1}{s\sqrt{s-a}} \exp\left[-\gamma_\Sigma s\right] p(\gamma_\Sigma)\, ds\, d\gamma_\Sigma \\ &= \frac{1}{2\pi}\int_1^\infty \frac{1}{s\sqrt{s-a}} \int_0^\infty \exp\left[-\gamma_\Sigma s\right] p(\gamma_\Sigma)\, d\gamma_\Sigma\, ds \\ &= \frac{1}{2\pi}\int_1^\infty \frac{C_{\bar{\gamma}_\Sigma}(s)}{s\sqrt{s-a}} ds.\end{aligned}$$

Let $\boldsymbol{\gamma} = \begin{bmatrix}\bar{\gamma}_1 & \bar{\gamma}_2 & \cdots & \bar{\gamma}_L\end{bmatrix}^T$. Define the L-variate MGF of $\boldsymbol{\gamma}$ as:

$$C_{\boldsymbol{\gamma}}(\mathbf{s}) = \int_0^\infty \cdots \int_0^\infty p_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}) \exp\left[-\mathbf{s}^T\boldsymbol{\gamma}\right] d\boldsymbol{\gamma} = \frac{1}{\left|I + D_s D_{\bar{\gamma}} M_x\right|}$$

for Rayleigh fading, where $\mathbf{s} = \begin{bmatrix}s_1 & s_2 & \cdots & s_L\end{bmatrix}^T$, I is an $L \times L$ identity matrix, D_s is a diagonal matrix $\mathrm{diag}\{s_1 \quad s_2 \quad \cdots \quad s_L\}$, $D_{\bar{\gamma}}$ is a diagonal matrix $\mathrm{diag}\{\bar{\gamma}_1 \quad \bar{\gamma}_2 \quad \cdots \quad \bar{\gamma}_L\}$, and M_X is the covariance matrix. For an array with L antennas equally spaced by distance d:

$$M_X(i,j) = \exp\left[-\frac{k}{2}(i-j)^2\left(\frac{d}{\lambda}\right)^2\right],\ i,j = 1,2,\cdots,L$$

where k is set to 21.4 to emulate the Bessel correlation model [10]. Next, $C_{\bar{\gamma}_\Sigma}(s)$ can be expressed in terms of $C_\gamma(\mathbf{s})$ through the following relationship:

$$C_{\bar{\gamma}_\Sigma}(s) = C_\gamma(\mathbf{s})\Big|_{s_1=s_2=\cdots=s_L=s}.$$

This gives us:

$$\boxed{\overline{P}_b = \frac{1}{2\pi}\int_1^\infty \frac{1}{s\sqrt{s-a}}\frac{1}{\left|I + sD_{\bar{\gamma}}M_X\right|}ds}.$$

3.3. Analytical BER Results

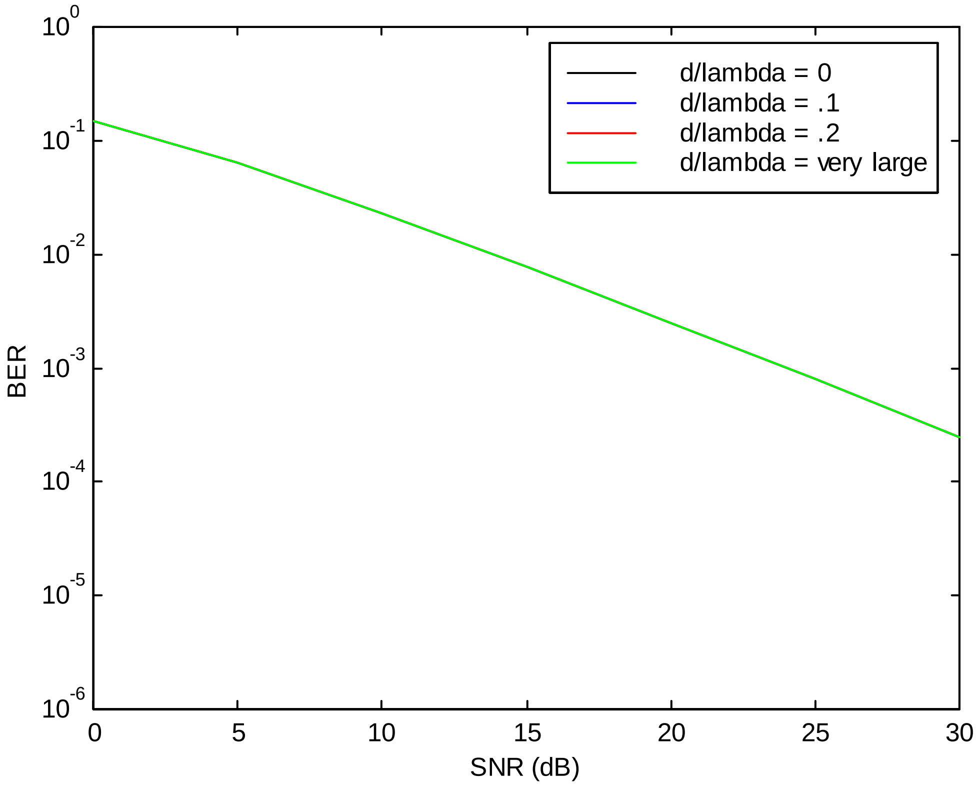

Figure 5: Analytical BER versus SNR for 1 receive antenna (L = 1)
(All lines overlap)

This result makes sense because for one receive antenna, the signal is correlated with itself.

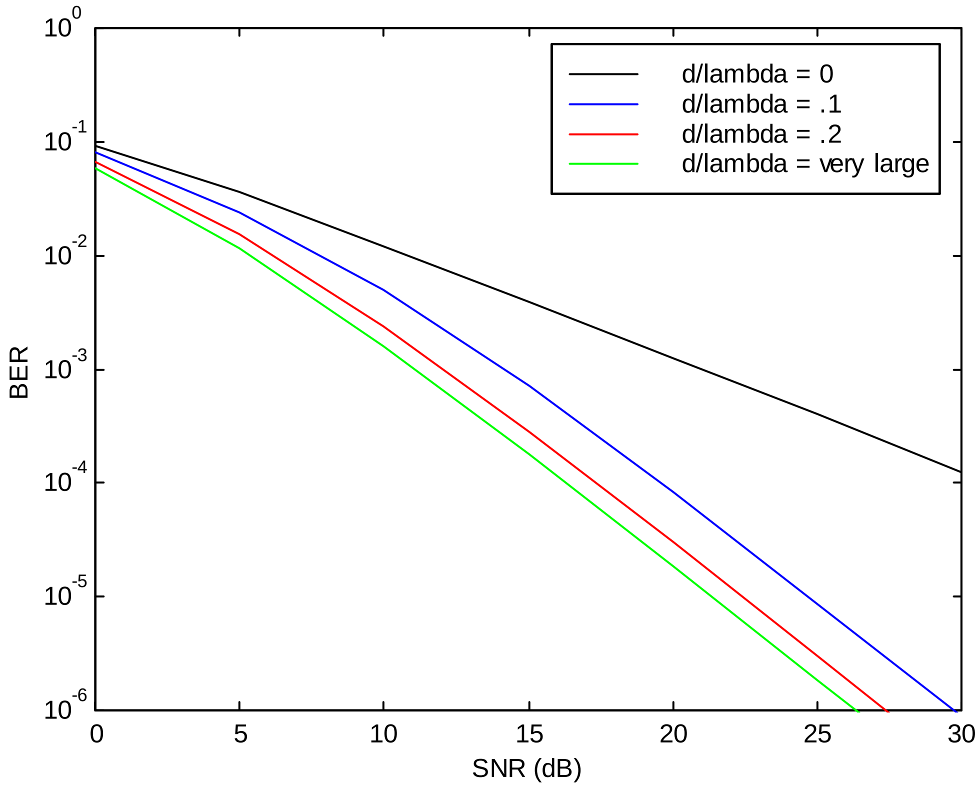

Figure 6: Analytical BER versus SNR for 2 receive antennas (L = 2)

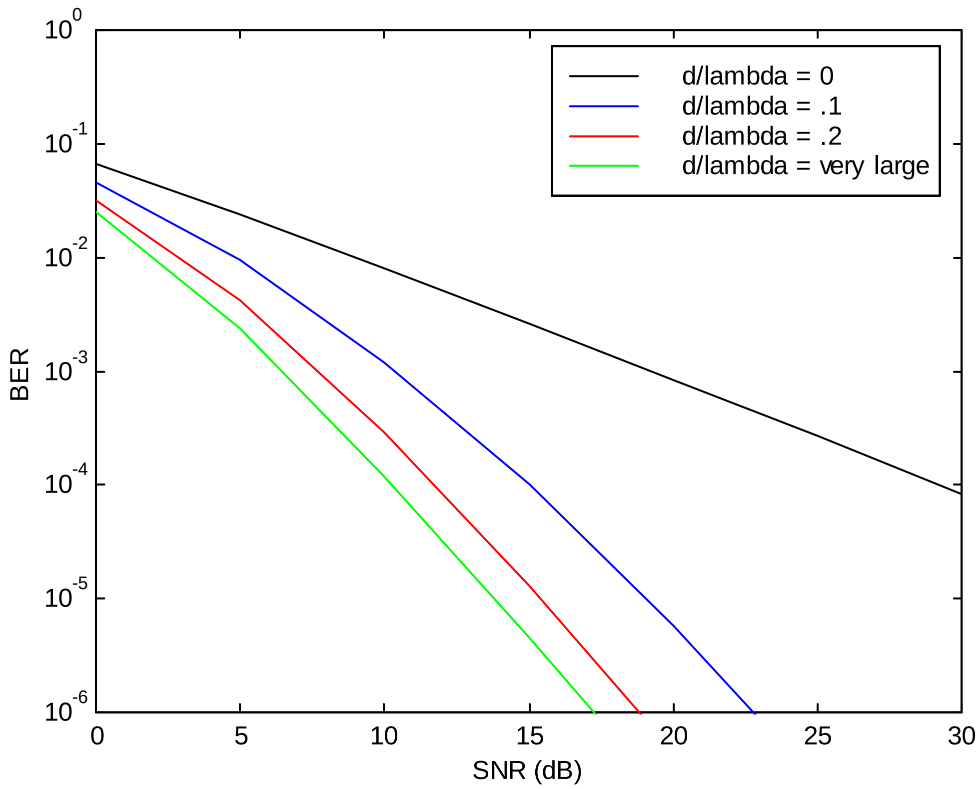

Figure 7: Analytical BER versus SNR for 3 receive antennas (L = 3)

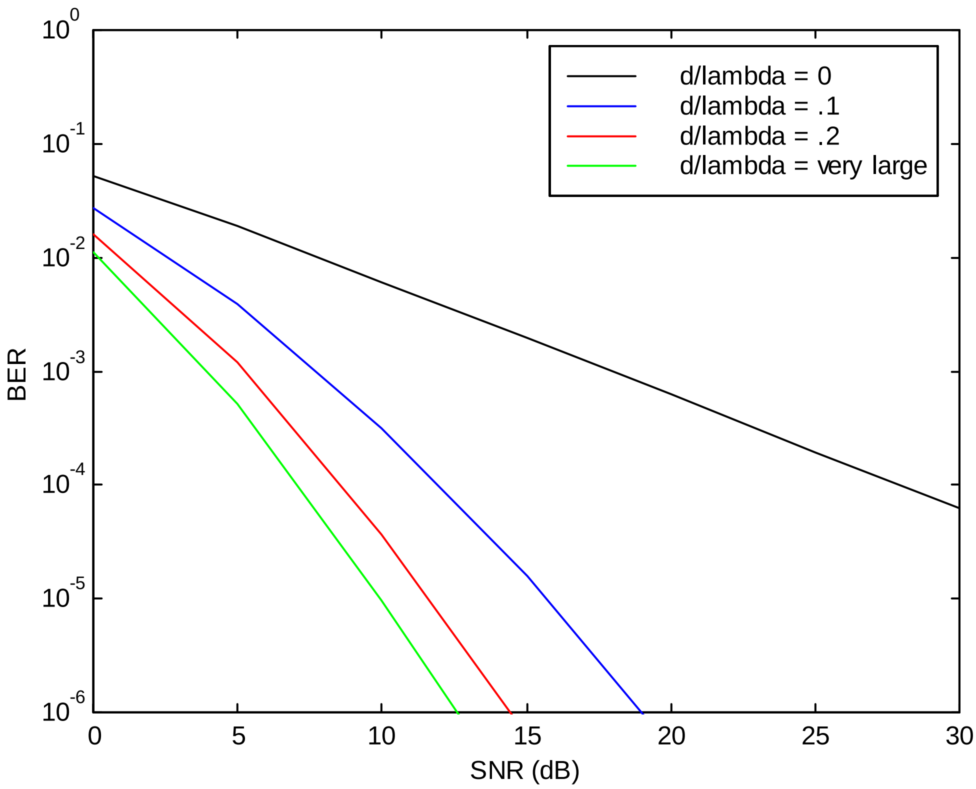

Figure 8: Analytical BER versus SNR for 4 receive antennas (L = 4)

From Figures 5 through 8 we see that if the antennas are spaced much less than $.1\lambda_c$ apart, then adding antennas does not improve BER performance significantly. However, for antenna separation distances greater than $.1\lambda_c$, adding antennas can significantly improve BER performance. We also notice that for a given number of receive antennas, increasing the separation distance improves BER performance as expected.

3.4. Simulation BER Results for MRC at the Mobile Station

In the simulations for correlated fading channels, we address only the case with two receive antennas. The correlation coefficient between the two fading channels is dependent on the separation distance between the two antennas. Our analytical expression for the probability of bit error was derived under the assumption of negligible scattering. This assumption holds true for MRC at the mobile station (MS) because the base station (BS) is typically a tall structure that rises above the surroundings such that there are no nearby scatterers [12]. Thus, we use Stuber's simple Bessel function model [13] for the correlation coefficient between received signals:

$$\rho = J_0^2\left(2\pi d / \lambda_c\right)$$

where $J_0(x)$ is the zeroth-order Bessel function of the first kind, d is the separation distance between the two antennas, and λ_c is the carrier wavelength.

MATLAB was used to compute the following results:

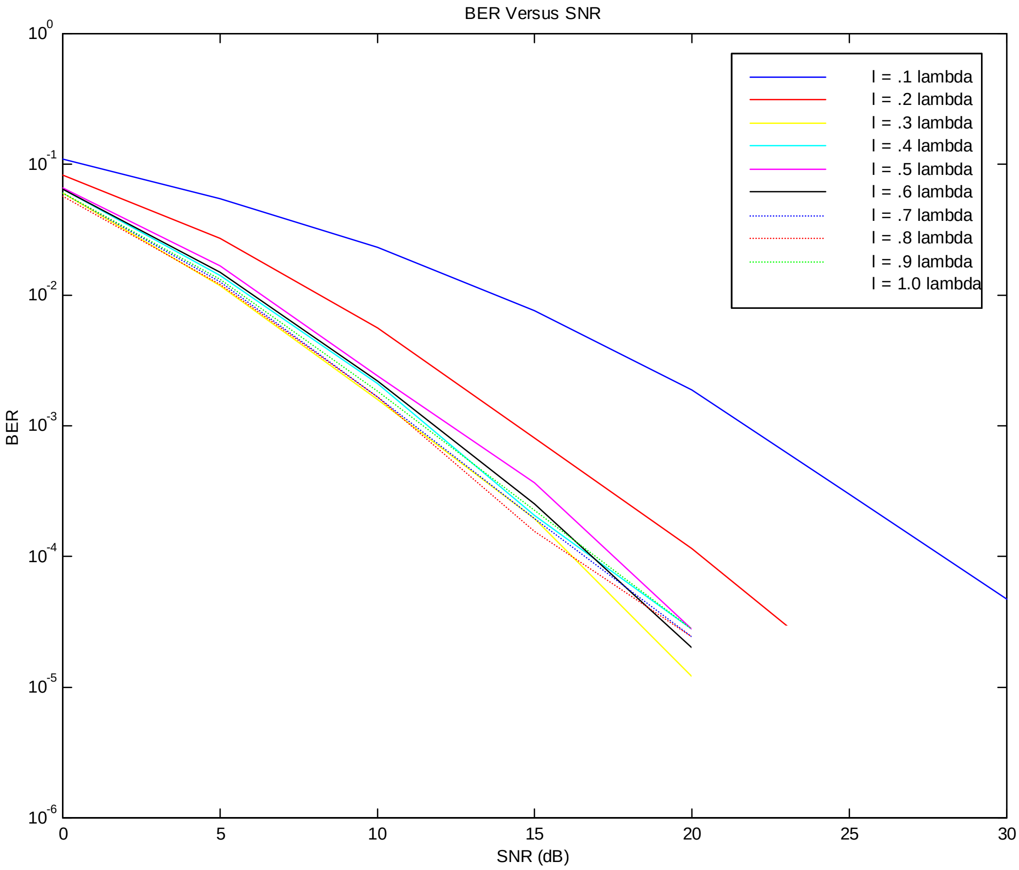

Figure 9: Simulation BER versus SNR for 2 receive antennas at the MS
(l = distance between antennas in the legend)

Observe that as $\frac{d}{\lambda_c}$ approaches .3 or .4, we have nearly optimal BER performance.

3.5. Simulation BER Results for MRC at the Base Station

Consider the following scattering model from [12]:

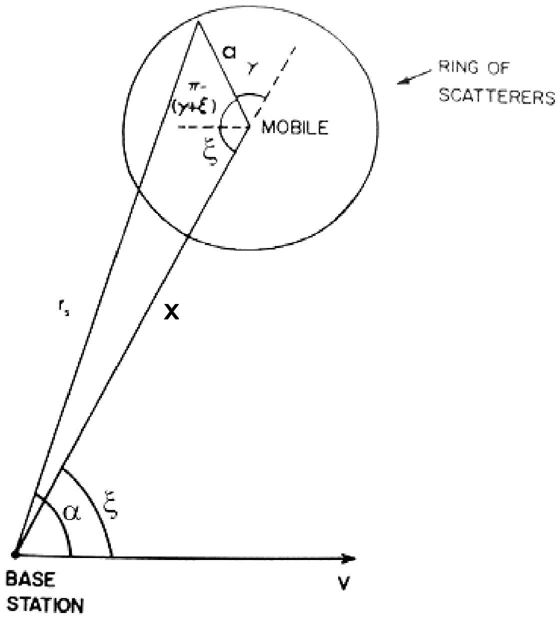

Figure 10: Effect of Scattering at the MS on the Received Signal at the BS [12]

In the scattering model, a denotes the scattering circle radius around the MS, x denotes the distance from the BS to the MS, and ξ denotes the angle in radians between the line-of-sight (LOS) and the direction of relative motion between the BS and the MS. For our purposes, we will assume that $\xi = \frac{\pi}{2}$ which means that the MS is moving perpendicular to the LOS. We will also assume that x is relatively large such that ξ does not change significantly during time intervals of interest and the ratio $k = \frac{a}{x}$ is very small. Assuming the MS to be operating at a point about 3.2 kilometers (about 2 miles) and a scattering radius of 16 meters (due to surrounding buildings), we get $k = .005$. The correlation coefficient using Jake's scattering model is given by [12]:

$$\boxed{\rho = J_0^2\left(2\pi\frac{d}{\lambda_c}k\sin\xi\right)J_0^2\left(\frac{1}{2}k^2 2\pi\frac{d}{\lambda_c}\sqrt{1-\tfrac{3}{4}\cos^2\xi}\right)}.$$

MATLAB was used to produce the following results:

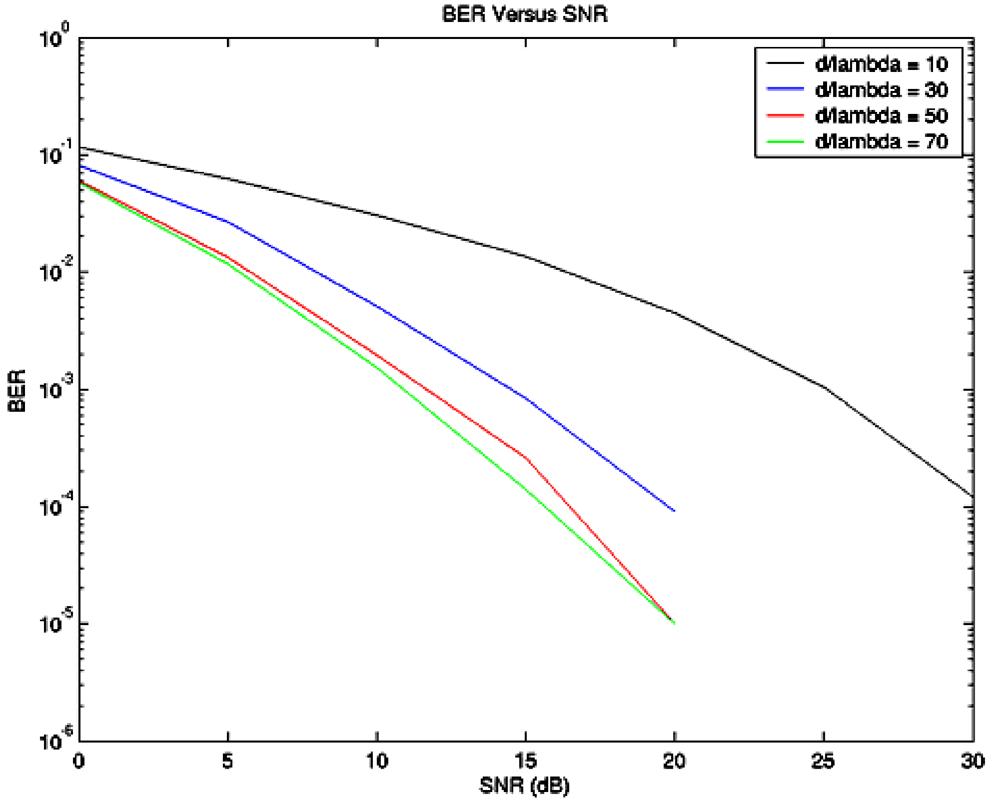

Figure 11: Simulation BER versus SNR for 2 receive antennas at the BS

3.6. Comparison of Simulation to Analytical Results

As our analytical results were based on negligible scattering, we first compare the simulation for MRC at the MS with the analytical results. Notice that the simulation result is slightly different from the analytical result, because Stuber's simple Bessel function model is meant as an approximation. The results depict BER improvement as antenna correlation decreases, but even with relatively high antenna correlation with $\frac{d}{\lambda_c} \approx .2 \Rightarrow \rho \approx .4128$, we still have considerable BER improvement with MRC at the MS. From Figure 9, we see that for $\frac{d}{\lambda_c} > .4 \Rightarrow \rho < .003021$ we achieve near-optimal BER performance.

Since the simulation results for MRC at the BS with scattering at the MS cannot be compared to the analytical results, we compare it to the simulation results for MRC at the MS. Significant improvement is observed when even with significant antenna correlation where $\frac{d}{\lambda_c} \approx 30 \Rightarrow \rho \approx 0.6240$. Notice that for the BS, the antennas must be separated much further apart from each other ($\frac{d}{\lambda_c} > 70 \Rightarrow \rho < 0.01229$) to achieve antenna decorrelation than for the MS. This is consistent with Jake's model where the BS antenna separation must be a factor of about $(k\sin\xi)^{-1} = \left(.005\sin\frac{\pi}{2}\right)^{-1} = 200$ times greater than the MS antenna separation in order to achieve the same fading channel decorrelation. This is intuitively satisfying because scatterers (such as those around the MS) would contribute to signal decorrelation, whereas the lack of scatterers (around the BS) would require a greater antenna separation distance to achieve the same degree of decorrelation.

4. CONCLUSION

We have achieved analytical and simulation results for the BER performance of maximal ratio combining under independent and correlated Rayleigh fading channels. At the mobile station, $\frac{d}{\lambda_c} > .4$ is sufficient antenna spacing to achieve decorrelated channels, while at the base station, $\frac{d}{\lambda_c}$ must be greater than 70 due to the signal scattering from the structures around the mobile station. Even if these antenna separation distances cannot be implemented in practice, performance can still be significantly improved with semi-correlated channels with ρ up to about .6.

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LIST OF VARIABLES FOR SECTION 2 (MRC with Independent Fading):

M = number of receive antennas

m = signal branch index

γ_Σ = SNR of post-combining signal

γ_m = SNR of signal branch m

r_m = fading envelope magnitude of signal branch m

θ_m = random phase of signal branch m

$s(t)$ = transmitted message signal

α_m = complex multiplier for signal branch m

a_m = amplitude of α_m

r_Σ = fading envelope magnitude of post-combining signal

N = noise power on each signal branch

N_Σ = noise power of post-combining signal

K = constant

P_b = probability of bit error

x = transmitted bit

$\mathbf{y}$ = received signal vector

$\mathbf{H}$ = channel matrix

$\mathbf{N}$ = noise vector

$\hat{x}$ = estimated bit

LIST OF VARIABLES FOR SECTION 3 (MRC with Correlated Fading):

γ_1, γ_2 = SNR of correlated fading signal branches

γ_3, γ_4 = SNR of independent fading signal branches

L = number of receive antenna branches

ℓ = signal branch index

$r_{\ell,I} + jr_{\ell,Q}$ = fading envelope expressed as sum of Gaussian in-phase and quadrature components

σ_G^2 = variance of Gaussian fading components

$y_1(t), y_2(t)$ = correlated received signal branches

$y_3(t), y_4(t)$ = independent received signal branches

ρ_I = correlation coefficient between in-phase fading components

ρ_Q = correlation coefficient between quadrature fading components

$\overline{\gamma}_\ell$ = average SNR of signal branch ℓ

T = transformation matrix

P_b = probability of bit error

$\overline{P}_b$ = average probability of bit error

γ_Σ = SNR of post-combining signal

d = antenna separation distance

$D_{\overline{\gamma}}$ = diagonal matrix of average SNRs on each branch

M_X = covariance matrix

$J_0(x)$ = zeroth-order Bessel function of the first kind

a = radius of scattering circle around mobile station (MS)

x = distance from base station (BS) to MS

k = ratio of a to x

ξ = angle in radians between line-of-sight (LOS) and direction of MS motion