Differential Modulation Using Space-Time Block Codes

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Abstract— **In this paper we consider a space-time differential modulation scheme where neither the transmitter nor the receiver has to know the channel. Our scheme is based on the theory of unitary space-time block codes. Compared to the existing differential modulation schemes for multiple antennas our scheme has a much smaller computational complexity. Moreover our codes have a higher coding gain and lower Bit Error Rate than the codes recently proposed by other researchers.**

I. Introduction

Using transmitter diversity for wireless communications has recently attracted a lot of attention. A simple transmitter diversity scheme using two transmitter antennas was proposed by Alamouti in [1]. An extension to more than two transmitter antennas was presented in [12] where it was shown that the Alamouti scheme is a special case of Space-Time Block Code (STBC). Exploiting transmitter diversity using amicable orthogonal designs was considered in [4]. The connection between the STBC of [12] and the approach in [4] was explored in [3]. Both approaches assume that the transmitter does not have channel state information (CSI) but the receiver needs CSI.

In this paper we describe a differential modulation scheme based on the codes proposed in [4]. Compared with the scheme proposed in [11] our scheme has a simpler transmitter and receiver. Compared to the schemes proposed in [8], [9] and [6] our scheme has a simpler decoder and for a given rate has a higher coding gain and a lower Bit Error Rate (BER).

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II. Channel Model

Consider a transmitter with n antennas and a receiver with m antennas. Let A be the $m \times n$ channel gain matrix. Thus the ij^{th} element of A is the (complex) gain factor for the path from the j^{th} transmit antenna to the i^{th} receive antenna. We assume that the received signal is corrupted by additive white Gaussian noise. Let W_t be the $n \times n$ matrix transmitted at a time t. Then the received $m \times n$ matrix R_t can be written as:

$$R_t = AW_t + \mathcal{N}_t \tag{1}$$

where $\mathcal{N}_t$ is the $m \times n$ white noise matrix whose elements are i.i.d Gaussian random variables with mean zero and variance σ^2.

The fading we consider here is frequency non-selective Rayleigh fading (or flat Rayleigh fading), in the case of which the elements of A are independent complex Gaussian random variables.

III. Unitary Codes Based on Amicable Orthogonal Designs

A. Amicable Orthogonal Designs

Let $\{X_j\}_{j=1}^p$ and $\{Y_j\}_{j=1}^p$ be a set of $2p$ matrices of size $n \times n$ which satisfy the following conditions:

$$\begin{array}{lll} X_jX_j^* = I & Y_jY_j^* = I & \forall j \\ X_jX_k^* = -X_kX_j^* & Y_jY_k^* = -Y_kY_j^* & \forall j \neq k \\ X_jY_k^* = Y_kX_j^* & & \forall j,k \end{array} \tag{2}$$

where I denotes the $n \times n$ identity matrix and $*$ denotes the conjugate transpose. Then $\{X_j\}$ and $\{Y_j\}$ are said to constitute an amicable orthogonal design of order n in p variables [4]. Explicit designs for $\{X_j\}$ and $\{Y_j\}$ (for $n = 2, 4, 8$) which satisfy the conditions in (2) are given in Appendix A. (Also see [4]).

B. Unitary Constellations

Let $\mathcal{S}$ denote the set of symbols from a (scalar) unitary constellation. That is, $s_j \in \mathcal{S}$ implies $|s_j|^2 = 1$. Well known examples of unitary constellations are BPSK, QPSK and 8-PSK. Let $\{s_j\}_{j=1}^p$ be a block of p symbols to be transmitted at a time t. Let s_j^R and s_j^I denote the real and imaginary parts of s_j, i.e, $s_j = s_j^R + i\, s_j^I$. Define

$$Z_t = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \left(X_j s_j^R + i\, Y_j s_j^I\right) \qquad (3)$$

Then it can be shown that:

$$\begin{aligned} Z_t Z_t^* &= \frac{1}{p}\left(\sum_{j=1}^{p} |s_j|^2\right) I_{n\times n} \\ &= \frac{1}{p}\, p\, I_{n\times n} = I_{n\times n} \end{aligned} \qquad (4)$$

Thus Z_t is a unitary code matrix. Note that Z_t is a Space-Time Block Code as defined in [12]. The connection between space-time block codes and amicable orthogonal designs is discussed in [3].

IV. Differential Detection

A. Transmission of square matrices

First we consider the case of $n \times n$ coding matrices. From the Appendix we can see that for $n = 2, 4$ and 8 we have such $n\times n$ matrices $\{X_j, Y_j\}$. The t^{th} block to be transmitted is an $n \times n$ matrix W_t. At the start of the transmission we transmit the $n \times n$ identity matrix. That is,

$$W_0 = I_{n\times n} \qquad (5)$$

Let $\{s_j\}_{j=1}^p$ be the set of p unitary symbols to be transmitted in the t^{th} block. Define, as in (3),

$$Z_t = \frac{1}{\sqrt{p}} \left(\sum_{j=1}^{p} X_j\, s_j^R + i \sum_{j=1}^{p} Y_j\, s_j^I\right) \qquad (6)$$

If W_{t-1} is the $(t-1)^{\text{th}}$ block then the t^{th} block transmitted is given by

$$W_t = W_{t-1} Z_t \qquad (7)$$

Assuming $W_{t-1}W_{t-1}^* = I_{n\times n}$, it follows from (6) and (3) that

$$W_t W_t^* = I_{n\times n} \qquad (8)$$

Since $W_0 W_0^* = I_{n\times n}$ we have:

$$W_t W_t^* = I_{n\times n} \quad \forall\, t \qquad (9)$$

To summarize the t^{th} block to be transmitted will be W_t given by (5), (6) and (7).

B. Maximum-Likelihood Detector

The received matrix at time t is given by:

$$\begin{aligned} R_t &= AW_t + \mathcal{N}_t \\ &= AW_{t-1} Z_t + \mathcal{N}_t \end{aligned} \qquad (10)$$

If AW_{t-1} were known to the receiver then the maximum likelihood (ML) detector for $\{s_j\}_{j=1}^p$ would be

$$\begin{aligned} \{\hat{s}_j\}_{j=1}^p = \text{Arg min}_{\{s_j\}, s_j\in\mathcal{S}}\; & tr\{(R_t - AW_{t-1}Z_t)^* \\ & \times (R_t - AW_{t-1}Z_t)\} \end{aligned} \qquad (11)$$

where $tr(\cdot)$ denotes the trace operator. The receiver can be simplified by noting that

$$\begin{aligned} & tr\left(\{R_t - AW_{t-1}Z_t\}^* \{R_t - AW_{t-1}Z_t\}\right) \\ &= tr\left(R_t^* R_t + Z_t^* W_{t-1}^* A^* A W_{t-1} Z_t \right. \\ &\quad \left. - R_t^* A W_{t-1} Z_t - Z_t^* W_{t-1}^* A^* R_t\right) \\ &= tr(R_t^* R_t) + tr(A^* A) \\ &\quad -2\, Real\{tr(R_t^* A W_{t-1} Z_t)\} \end{aligned} \qquad (12)$$

Thus the ML detector for $\{s_j\}_{j=1}^p$ becomes

$$\begin{aligned} \{\hat{s}_j\}_{j=1}^p &= \text{Arg max}_{\{s_j\}, s_j\in\mathcal{S}}\; Real\{tr(R_t^* A W_{t-1} Z_t)\} \\ &= \text{Arg max}_{\{s_j\}, s_j\in\mathcal{S}} \sum_{j=1}^{p} \{\, Real\{tr(R_t^* A W_{t-1} X_j)\} s_j^R \\ &\qquad + Real\{tr(R_t^* A W_{t-1}\, i\, Y_j)\} s_j^I \} \end{aligned} \qquad (13)$$

Hence the maximum likelihood detector for s_j is given by

$$\begin{aligned} \hat{s}_j &= \text{Arg} \max_{s_j \in \mathcal{S}} \left[Real\{tr(R_t^* A W_{t-1} X_j)\} s_j^R \right. \\ &\qquad \left. + Real\{tr(R_t^* A W_{t-1}\, i\, Y_j)\} s_j^I\right] \end{aligned} \qquad (14)$$

In the differential case we assume that the receiver does not know the channel and hence does not know AW_{t-1}. The received signal at time $t-1$ was

$$R_{t-1} = AW_{t-1} + \mathcal{N}_{t-1} \qquad (15)$$

Since $\mathcal{N}_{t-1}$ is a Gaussian white noise, R_{t-1} can be taken as the Maximum Likelihood estimate of AW_{t-1} (based on one block). Substituting R_{t-1} for AW_{t-1} in (14) we get the expression for the Maximum Likelihood detector: [1]

$$\begin{aligned} \hat{s}_j &= \text{Arg} \max_{s_j \in \mathcal{S}} \left[Real\{tr(R_t^* R_{t-1} X_j)\} s_j^R \right. \\ &\qquad \left. + Real\{tr(R_t^* R_{t-1}\, i\, Y_j)\} s_j^I\right] \end{aligned} \qquad (16)$$

[1] A more rigorous proof is provided in [2]

Note that the detector has a decoupled form: one scalar detector for each of the symbols $\{s_j\}$. Compared to the detectors in [6], [8] and [11], the above detector has a much lower computational complexity. It is shown in [2] that the SNR for the differential detector in (16) is approximately given by

$$\mathrm{SNR}_{diff} \approx \frac{tr(A^*A)}{2\sigma^2}E_s \quad (17)$$

which is about 3 dB lower than the SNR for the coherent detector in (14). This is in agreement with what is known for the conventional DPSK scheme for a single antenna system. (See [10] for details). Since the elements of A are i.i.d. complex Gaussian random variables, the SNR is a chi-squared random variable with $2mn$ degrees of freedom. Thus a diversity of order mn is achieved, like in the coherent detection case.

C. Transmission of wide matrices

The scheme outlined in the previous sections is valid for $n = 2, 4$ and 8 transmitter antennas. Now we consider the extension of the scheme for $n = 3, 5, 6$ and 7 transmitter antennas. Let $W_t^{(4)}$ and $W_t^{(8)}$ denote the unitary matrices W_t designed for 4 and 8 transmitter antennas respectively.

For $n = 3$ transmitter antennas, we transmit $\Phi_3 W_t^{(4)}$ as the t^{th} block where

$$\Phi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (18)$$

The multiplication factor Φ_3 can be absorbed in the channel matrix. The received matrix R_t can be written as:

$$\begin{aligned} R_t &= A\Phi_3 W_t^{(4)} + \mathcal{N}_t \\ &= \bar{A} W_t^{(4)} + \mathcal{N}_t \end{aligned} \quad (19)$$

where

$$\bar{A} = A\Phi_3 \quad (20)$$

From (19) we can see that at the receiving side we can use the ML detector in (16). Since Φ_3 satisfies the condition $\Phi_3\Phi_3^* = I_{3\times 3}$ we have that

$$\bar{A}\bar{A}^* = AA^*$$

and hence

$$tr(\bar{A}^*\bar{A}) = tr(A^*A)$$

Hence we achieve the same type of diversity and SNR performance as for the other values of n. Note that transmitting $\Phi_3 W_t^{(4)}$ is equivalent to transmitting the first three rows of $W_t^{(4)}$.

For the case of $n = 5, 6, 7$ transmitter antennas we use an analogous scheme. We transmit $\Phi_n W_t^{(8)}$ where Φ_n is an $n \times 8$ matrix given by

$$\Phi_n = \left[I_{n\times n}\, \mathbf{0}_{n\times(8-n)} \right] \quad (21)$$

where $I_{n\times n}$ is the $n \times n$ identity matrix and $\mathbf{0}$ is an all-zero matrix. Along the lines of reasoning used for $n = 3$ we can show that the same type of SNR and diversity performance are obtained. Transmitting $\Phi_n W_t^{(8)}$ is of course equivalent to transmitting the first n rows of $W_t^{(8)}$.

V. Coding Gain Comparisons

The differential space-time coding schemes proposed in [6] and [8] are based on using unitary matrices as symbols of a constellation. This can be thought of as the matrix counterpart of the unitary scalar constellations. In [8] and [6] the design of a set of $n \times n$ unitary code matrices $\mathcal{G}$ is considered such that $\mathcal{G}$ has a group structure.

For a scalar constellation a good metric to judge the performance is the square of the minimum distance between two points in the constellation, that is,

$$\min_{s_k, s_l \in \mathcal{S}; k\neq l} |s_l - s_k|^2 \quad (22)$$

Based on the analysis of error probability it was shown in [8] that a good metric to judge the performance of a unitary matrix constellation is the *coding gain*. The coding gain (as defined in [8]) is:

$$\min_{C_k, C_l \in \mathcal{G}; k\neq l} n\,\Lambda(C_k, C_l) \quad (23)$$

where

$$\Lambda(C_k, C_l) = |(C_k - C_l)(C_k - C_l)^*|^{1/n}$$

and $|\cdot|$ denotes the determinant.

As the code matrices $\{Z_t\}$ we considered are also unitary the same criterion can be used to judge their performance . Because $\{Z_t\}$ are constructed as a linear combination of the symbols from $\mathcal{S}$ it is natural that their performance depends on the choice of $\mathcal{S}$. Let Z_k be the unitary matrix constructed from the symbols $s_k(1), \ldots, s_k(p)$, where $\{s_k(j)\} \in \mathcal{S}$. Let Z_l be another unitary matrix constructed from the set of symbols $s_l(1), \ldots, s_l(p)$. Proceeding in a way similar to (3) it can be shown that:

$$(Z_k - Z_l)(Z_k - Z_l)^* = \frac{1}{p}\left(\sum_{j=1}^{p} |s_k(j) - s_l(j)|^2\right) I_{n\times n}$$

and hence

$$|(Z_k-Z_l)(Z_k-Z_l)^*|^{1/n} = \frac{1}{p}\left(\sum_{j=1}^{p}|s_k(j)-s_l(j)|^2\right)$$

For $Z_k \neq Z_l$, the above quantity is minimized when $\{s_k(j)\}$ and $\{s_l(j)\}$ differ in just one symbol. The minimal value is

$$\frac{1}{p}|s_l - s_k|^2$$

where we have dropped the index j. Therefore the coding gain is given by

$$\min_{s_k,s_l\in\mathcal{S};k\neq l}(n/p)|s_l-s_k|^2 \tag{24}$$

Optimal unitary codes are discussed in [6], [8] and [9]. However they apply only to group codes. In general $\{Z_t\}$ does not have a group structure and hence does not fall in the class of codes considered in [6], [8] and [9].

When we compare the codes proposed in this paper and the codes proposed in [6], [8] we must take into account the spectral efficiency of the code. Spectral efficiency is the number of bits transmitted per second/cycle. Depending on the choice of $\mathcal{S}$ we get different spectral efficiencies for $\{Z_t\}$. We give in Table I the spectral efficiencies and the corresponding coding gains for QPSK and 8-PSK constellations. For comparison we have also shown the coding gain for the codes in [8], [9]. Whenever the code corresponding to a certain spectral efficiency has not been reported in [8], [9], we have left the corresponding entry blank.

From the table we can see that for a given spectral efficiency our differential scheme has a better coding gain than the one in [8], [9]. Moreover since our scheme is based on Space-Time block codes, it is also computationally more attractive.

TABLE I

CODING GAINS FOR TWO, THREE AND FOUR TRANSMITTER ANTENNAS. THE RATE IS GIVEN IN BITS/SEC/HZ.

Number of Antennas	Symbol Set	Rate	Coding Gain	
			New Codes	Codes in [8], [9]
2	QPSK	2	2	1.531
	8-PSK	3	0.5858	-
3	QPSK	1.5	2	-
	8-PSK	2.25	0.5858	-
4	QPSK	1.5	2.7	1.85
	8-PSK	2.25	0.78	-

VI. Numerical Studies

In this section we present numerical examples for our differential detection scheme and compare its performance with that of the coherent detection scheme in terms of the Bit Error Rate (BER). The channel we consider is a flat Rayleigh fading channel. The elements of A, $\{A_{ij}\}$, are considered to be i.i.d. complex Gaussian random variables with mean zero and variance equal to one: $A_{ij} \sim \mathcal{N}(0,1)$. We consider a system with two transmitter antennas and one receiver antenna. The signal power is set to unity: $E_s = 1$. The variance of the additive Gaussian white noise is varied to obtain different SNR values. For the coherent detection scheme we assume that perfect CSI is available at the receiver. The Bit Error rates were obtained from 10^6 Monte-Carlo simulation runs. Both A and the noise realizations were varied.

In Figure 1 we compare the differential detection scheme using QPSK modulation (and hence having a spectral efficiency of 2 bits/sec/Hz; see Table I) with the optimal code proposed in [8] having the same spectral efficiency. From the figure we can see that the proposed differential Space-Time block code outperforms the differential code in [8] by about 3 dB. It can also be seen that the coherent detector outperforms the corresponding differential detector by 3 dB. However to implement the coherent detector the reciever need to know the channel.

VII. Conclusion

In this paper we considered the use of Unitary Space-Time Block Codes for differential modulation. We showed that their use, when compared to other existing codes, leads to a computationally simpler receiver and also that for a given rate they have a higher coding gain. More details of these codes and their performance analysis will be reported in [2].

Appendix A

Here we present the explicit design of $\{X_j\}$ and $\{Y_j\}$ for $n=2$ as well as the procedure for construction of these coding matrices for $n=4$ and $n=8$. The reader interested in more details is referred to [5].

Design for $n=2$

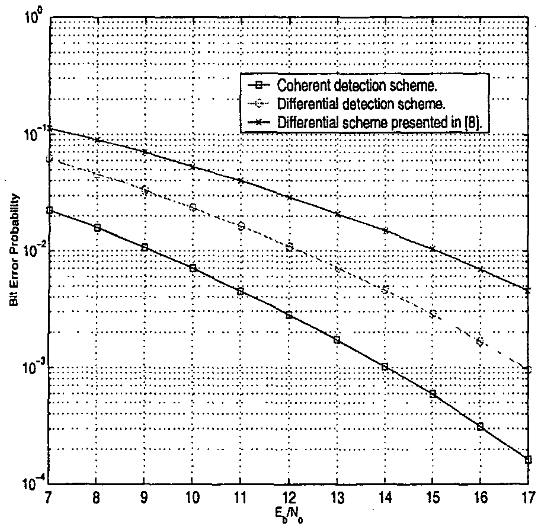

Fig. 1. Bit Error Rates for a Rate of 2 bits/sec/Hz.

Consider the 2×2 matrices:

$$X_1^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X_2^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$Y_1^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y_2^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (A.1)$$

The reader can verify that the matrices in (A.1) satisfy the conditions in (2).

Design for $n = 4$

Define the following matrices:

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (A.2)$$

If C and D are two matrices then $C \otimes D$ denotes the Kronecker (tensor) product of C and D [7]. Let I_n denote the identity matrix of size $n \times n$. Then a set of three 4×4 matrices $\{X_j^{(4)}\}$ and $\{Y_j^{(4)}\}$ which satisfy the conditions in (2) are given by:

$$\begin{aligned} X_1^{(4)} &= I_4 \\ X_2^{(4)} &= P \otimes X_2^{(2)} \\ X_3^{(4)} &= T \otimes I_2 \\ Y_j^{(4)} &= P \otimes Y_j^{(2)} \quad j = 1,2 \\ Y_3^{(4)} &= Q \otimes I_2 \end{aligned} \qquad (A.3)$$

The reader familiar with the theory of amicable orthogonal designs will note that what we have used to obtain (A.3) is Wolfe's Slide Lemma [5].

Design for $n = 8$

A set of four 8×8 matrices $\{X_j^{(8)}\}$ and $\{Y_j^{(8)}\}$ which satisfy the conditions in (2) are given by:

$$\begin{aligned} X_1^{(8)} &= I_8 \\ X_j^{(8)} &= P \otimes X_j^{(2)} \quad j = 2,3 \\ X_4^{(8)} &= T \otimes I_4 \end{aligned} \qquad (A.4)$$

$$\begin{aligned} Y_j^{(8)} &= P \otimes Y_j^{(4)} \quad j = 1,2,3 \\ Y_4^{(8)} &= Q \otimes I_4 \end{aligned} \qquad (A.5)$$

where the matrices $\{X_j^{(4)}\}$ and $\{Y_j^{(4)}\}$ are as given in (A.3). This construction is also based on Wolfe's Slide Lemma.

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