models may also be justified by virtue of a fundamental theorem of time series analysis, which is discussed next.

2.6 WOLD DECOMPOSITION

Wold (1938) proved a fundamental theorem, which states that any stationary discrete-time stochastic process may be decomposed into the sum of a *general linear process* and a *predictable process*, with these two processes being uncorrelated with each other. More precisely, Wold proved the following result:

Any stationary discrete-time stochastic process $x(n)$ may be expressed in the form

$$x(n) = u(n) + s(n) \tag{2.54}$$

where

1. $u(n)$ and $s(n)$ are uncorrelated processes,
2. $u(n)$ is a general linear process represented by the MA model:

$$u(n) = \sum_{k=0}^{\infty} b_k^* v(n-k) \tag{2.55}$$

with $b_0 = 1$, and

$$\sum_{k=0}^{\infty} |b_k|^2 < \infty,$$

and where $v(n)$ is a white-noise process uncorrelated with $s(n)$; that is,

$$E[v(n)s^*(k)] = 0 \qquad \text{for all } (n, k)$$

3. $s(n)$ is a predictable process; that is, the process can be predicted from its own past with zero prediction variance.

This result is known as *Wold's decomposition theorem*. A proof of this theorem is given in Priestley (1981).

According to Eq. (2.55), the general linear process $u(n)$ may be generated by feeding an *all-zero filter* with the white-noise process $v(n)$ as in Fig. 2.5(a). The zeros of the transfer function of this filter equal the roots of the equation:

$$B(z) = \sum_{n=0}^{\infty} b_n^* z^{-n} = 0$$

A solution of particular interest is an all-zero filter that is *minimum phase*, which means that all the zeros of the polynomial $B(z)$ lie inside the unit circle. In such a case, we may replace the all-zero filter with an *equivalent* all-pole filter that has the same impulse response $h_n = b_n^*$, as in Fig. 2.5(b). This means that except for a predictable component, a stationary discrete-time stochastic process may also be represented as an AR process of the appropriate order, subject to the above-mentioned restriction on $B(z)$. The basic difference between the MA and AR models is that $B(z)$ operates on the input $v(n)$ in the MA model, whereas the inverse $B^{-1}(z)$ operates on the output $u(n)$ in the AR model.

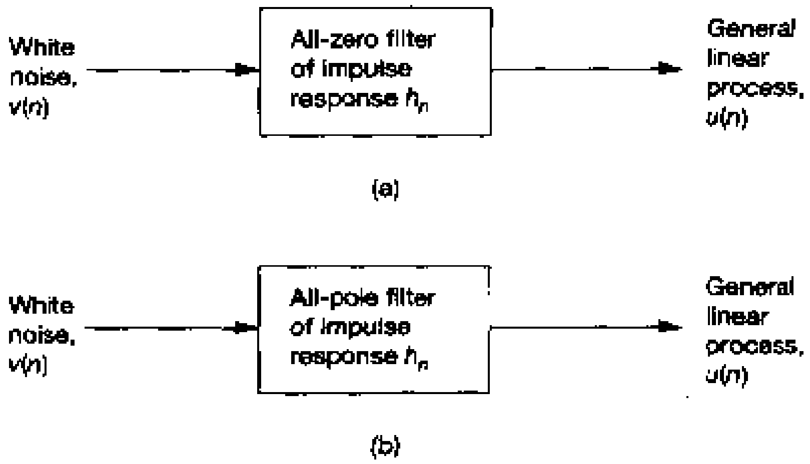

Figure 2.5 (a) Model, based on all-zero filter, for generating the linear process $u(n)$; (b) model, based on all-pole filter, for generating the general linear process $u(n)$. Both filters have exactly the same impulse response.

2.7 ASYMPTOTIC STATIONARITY OF AN AUTOREGRESSIVE PROCESS

Equation (2.42) represents a *linear, constant coefficient, difference equation of order M*, in which $v(n)$ plays the role of *input* or *driving function* and $u(n)$ that of *output* or *solution*. By using the *classical method*[1] for solving such an equation, we may formally express the solution $u(n)$ as the sum of a *complementary function*, $u_c(n)$, and a *particular solution*, $u_p(n)$, as follows:

$$u(n) = u_c(n) + u_p(n) \tag{2.56}$$

The evaluation of the solution $u(n)$ may thus proceed in two stages:

1. The complementary function $u_c(n)$ is the solution of the *homogeneous equation*

$$u(n) + a_1^* u(n-1) + a_2^* u(n-2) + \cdots + a_M^* u(n-M) = 0$$

   In general, the complementary function $u_c(n)$ will therefore be of the form

$$u_c(n) = B_1 p_1^n + B_2 p_2^n + \cdots + B_M p_M^n \tag{2.57}$$

   where $B_1, B_2, \ldots, B_M$ are arbitrary constants, and $p_1, p_2, \ldots, p_M$ are roots of the characteristic equation (2.51).

2. The particular solution $u_p(n)$ is defined by

$$u_p(n) = H_G(D)[v(n)] \tag{2.58}$$

[1]We may also use the z-transform method to solve the difference equation (2.42). However, for the discussion presented here, we find it more informative to use the classical method