Codes for Differential Signals in the Presence of Large Numbers of Antennas

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Abstract

We consider the design of codes for differential space-time modulation in the presence of large numbers of transmit and/or receive antennas. Based on the novel upper bound on the pairwise-error probability of differential space-time modulation for large numbers of antennas, we show that Euclidean distance is an appropriate code performance indicator in the large-array regime. For two transmit antennas, we use the new design criterion to obtain some new differential codes with large minimum Euclidean distance. Simulations of bit-error-rate confirm that the new codes improve the performance of differential codes for large numbers of receive antennas.

1 Introduction

Differential space-time modulation (DSTM), which doesn't require channel estimates at the transmitter or receiver, has gained much attention in recent years ([7, 3, 2], etc).

In [3, 2], it was proposed that the error probability of differential space-time codes on quasi-static flat fading channels can be made small at high signal-to-noise ratios (SNRs) by designing codes according to the rank and determinant criteria — the same design criteria proposed earlier for coherent space-time codes [6]. More recently, Biglieri *et al* [1] and Yuan *et al* [9] have suggested that Euclidean distance is actually a better predictor of *coherent* space-time code performance when the number of transmit and/or receive antennas is large and SNR is moderate. It is natural to ask whether this observation also applies to differential space-time modulation.

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In this paper, we consider the design and analysis of differential space-time modulation for large numbers of transmit and/or receive antennas. Based on the novel upper bound on the pairwise-error probability of DSTM in the presence of large numbers of antennas, we show that Euclidean distance is an appropriate code performance indicator in the large-array regime. For two transmit antennas and many receive antennas, we then use the new design criterion to obtain some new differential codes with large minimum Euclidean distance. Simulations of bit-error-rate show that the new codes improve the performance of differential codes for four or more receive antennas.

The rest of the paper is organized as follows. In Sec. 2, we introduce the channel model and review the general structure of DSTM. The pairwise upper bound and design criterion for quasi-static Rayleigh fading channel are derived in Sec. 3. In Sec. 4, some new differential codes for two transmit antennas are constructed. Sec. 5 presents the performance simulation results. Our main conclusions are summarized in Sec. 6.

2 Differential modulation

Consider a wireless channel in which data are sent from t transmit antennas to r receive antennas. Under slowly-varying, flat fading conditions, the channel can be modeled by

$$Y = \sqrt{\rho/t}HX + N \qquad (1)$$

where X is a $t \times t$ matrix of transmitted signals, Y is an $r \times t$ matrix of received signal samples, H is an $r \times t$ matrix of fading path gains, and N is an $r \times t$ matrix of noise. We assume that the elements of H and N are independent, identically distributed (i.i.d.) complex Gaussian random variables $\mathcal{CN}(0, 1)$. We further restrict attention to signal matrices that satisfy

$$XX^{\dagger} = tI_t \tag{2}$$

where I_t is the $t \times t$ identity matrix and $\dagger$ is the conjugate-transpose. Consequently, ρ represents the signal-to-noise-ratio (SNR) per receiver antenna.

In differential space-time modulation (DSTM), the transmitted signal matrix for block k is given by

$$X_k = X_{k-1}G_k, \ k = 1, \ldots, K. \tag{3}$$

where G_k is taken from a unitary matrix constellation $\mathcal{G}$ and X_0 is any fixed matrix satisfying (2). The corresponding data rate is then $R = (1/t)\log_2 |\mathcal{G}|$ bps/Hz.

If H is approximately constant for $2t$ channel symbols, the signal blocks can be detected without channel estimates using a simple differential receiver [3, eq. 15]

$$\hat{G}_k = \arg \max_{G \in \mathcal{G}} \mathrm{ReTr}\{GY_k^{\dagger}Y_{k-1}\} , \tag{4}$$

where "ReTr" denotes the real part of the matrix trace.

For example, one simple unitary constellation $\mathcal{G}$ — equivalent to the scheme proposed by Tarokh and Jafarkhani [7] — consists of all matrices of the form

$$G = \frac{1}{\sqrt{2}} \begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix}, \tag{5}$$

where x, y belong to the unit-energy constellation $\{\exp(2\pi ji/M) : i = 0, \ldots, M-1\}$. Other examples are the differential group codes in [3, 2], where the constellation $\mathcal{G}$ forms a group under matrix multiplication.

3 Error bound and design criteria

Previous results on error bounds for DSTM [7, 3, 2] have demonstrated that, in the high-SNR limit, performance is determined primarily by the same criteria as in coherent space-time modulation: the rank and determinant criteria [6] applied to the constellation $\mathcal{G}$. For large r and t, however, error probabilities of practical interest are often achieved by modest SNRs, where the rank and determinant criteria do not apply. Consequently, our aim in this section is to develop new design criteria applicable to large t and r, and moderate SNR. To this end, we begin by deriving a bound on the pairwise error probability, conditioned on the current channel fading matrix H.

For the differential receiver (4), the pairwise error probability of transmitting matrix G and erroneously decoding $\hat{G}$, conditioned on H, can be written as

$$\Pr(G \to \hat{G}|G, H) = \Pr(\Delta \geq 0|G, H) , \tag{6}$$

where

$$\Delta = \mathrm{ReTr}\{(\hat{G} - G)Y_k^{\dagger}Y_{k-1}\}.$$

By ignoring the second-order noise term in Δ, (6) is approximated by

$$\begin{aligned} \Pr(G \to \hat{G}|G, H) &= \Pr(\Delta \geq 0|G, H) \\ &\approx Q\left\{\frac{1}{2} \cdot \sqrt{\frac{\rho}{t}||HX_{k-1}(G - \hat{G})||^2}\right\} \\ &\leq \frac{1}{2}\exp\left\{-\frac{\rho}{8}||H'(G - \hat{G})||^2\right\} \end{aligned} \tag{7}$$

where $||\cdot||$ is the Frobenius norm and $H' = HX_{k-1}/\sqrt{t}$ follows the same distribution as H.

For fixed t, it is shown in [1] that $\| H'(G-\hat{G}) \|^2 /r$ converges almost surely as $r \to \infty$ to $\| G - \hat{G} \|^2$. The same is shown for $r, t \to \infty$ with $r/t \to c > 0$. In [9], the authors assert without proof that the exponent above converges to a Gaussian random variable as $r \cdot \nu \to \infty$, where $\nu = \mathrm{rank}(G - \hat{G})$. For fixed ν and $r \to \infty$, this is clearly true; however, for fixed r and $\nu \to \infty$, it is not difficult to contrive counterexamples. It therefore appears that additional conditions on $G - \hat{G}$ are needed. We now show that if G and $\hat{G}$ are *unitary*, and the distance per dimension is bounded away from zero

$$\frac{1}{\nu} \| G - \hat{G} \|^2 \geq d > 0 , \tag{8}$$

then convergence of the (suitably normalized) exponent to a Gaussian distribution is assured as $\nu \to \infty$.

To this end, define the $t \times t$ code distance matrix

$$A(G, \hat{G}) = (G - \hat{G})(G - \hat{G})^{\dagger} , \tag{9}$$

the bound (7) can then be rewritten as

$$\Pr(G \to \hat{G}|G, H) \leq \frac{1}{2}\exp\left(-\frac{\rho}{8}\sum_{i=1}^{r}\sum_{j=1}^{\nu}\lambda_j U_{ij}\right) . \tag{10}$$

where $\lambda_j > 0, j = 1, 2, \ldots, \nu$ are the non-zero eigenvalues of $A(G, \hat{G})$ and U_{ij} are i.i.d. exponential random variables with probability density function (pdf) $p(t) = e^{-t}u(t)$, and mean and variance 1.

Now consider the normalized sum

$$S_{r\nu} = \sum_{i=1}^{r}\sum_{j=1}^{\nu} Z_{ij} \ \text{ where } \ Z_{ij} = \frac{\lambda_j(U_{ij} - 1)}{\sqrt{r\sum_{j=1}^{\nu}\lambda_j^2}} . \tag{11}$$

Observe that Z_{ij} has mean zero and variance

$$\sigma_{ij}^2 = \frac{\lambda_j^2}{r\sum_{j=1}^{\nu}\lambda_j^2} \ \text{ where } \ \sum_{i=1}^{r}\sum_{j=1}^{\nu}\sigma_{ij}^2 = 1 .$$

Let $\xrightarrow{d}$ denote convergence in distribution. By the Lindeberg-Feller version of the Central Limit Theorem [5, pp. 326], a sufficient condition for $S_{r\nu} \xrightarrow{d} \mathcal{N}(0,1)$ as $r\nu \to \infty$ is the *Lindeberg condition*: For all $\epsilon > 0$, we have

$$\lim_{r\nu\to\infty} \sum_{i=1}^{r}\sum_{j=1}^{\nu} \mathcal{E}\left[Z_{ij}^2 I_{[\epsilon,\infty)}\left(|Z_{ij}|\right)\right] \;=\; 0$$

where $I_{[\epsilon,\infty)}(x)$ is the indicator function and $\mathcal{E}$ means averaging with respect to Z_{ij}. Substituting Z_{ij} into the sum above, we obtain

$$\begin{aligned}
&\sum_{i=1}^{r}\sum_{j=1}^{\nu}\sigma_{ij}^2 \int_{|\sigma_{ij}(t-1)|>\epsilon} (t-1)^2 e^{-t} u(t)dt \\
&\leq\; 2e\sum_{i=1}^{r}\sum_{j=1}^{\nu}\sigma_{ij}^2 \int_{\epsilon/\sigma_{ij}}^{\infty} y^2 e^{-y} dy \\
&\leq\; \frac{26\sum_{j=1}^{\nu}\lambda_j^3}{\epsilon\sqrt{r}(\sum_{j=1}^{t}\lambda_j^2)^{\frac{3}{2}}}\,. \qquad (12)
\end{aligned}$$

For unitary matrices G and $\hat{G}$, it is easy to verify that each λ_j is bounded between 0 and 4. Combining this fact with the convexity of $f(x) = x^2$, we obtain

$$\frac{26\sum_{j=1}^{\nu}\lambda_j^3}{\epsilon\sqrt{r}(\sum_{j=1}^{\nu}\lambda_j^2)^{\frac{3}{2}}} \;\leq\; \frac{26\cdot 4}{\epsilon\sqrt{r}(\sum_{j=1}^{\nu}\lambda_j^2)^{\frac{1}{2}}} \leq \frac{104}{\epsilon d\sqrt{r\nu}}\,,$$

from which we conclude that $S_{r\nu}$ satisfies the Lindeberg condition for $r \to \infty$ or $\nu \to \infty$. Therefore, $S_{r\nu} \xrightarrow{d} \mathcal{N}(0,1)$ as $r\nu \to \infty$. It follows that the quantity $(1/r)\sum_{i=1}^{r}\sum_{j=1}^{\nu}\lambda_j U_{ij}$ approaches a Gaussian random variable D with $\mathcal{N}(\mu_D, \sigma_D^2)$ where

$$\mu_D = \|G-\hat{G}\|^2 = \sum_{j=1}^{\nu}\lambda_j \quad\text{and}\quad \sigma_D^2 = \sum_{j=1}^{\nu}\lambda_j^2/r$$

The differential pairwise error probability is then bounded by

$$\begin{aligned}
P(G\to\hat{G}) &\leq \int_{-\infty}^{+\infty}\frac{1}{2}\exp(-\frac{\rho r}{8}D)p(D)dD \\
&= \frac{1}{2}\exp(-\frac{\rho r}{8}\sum_{j=1}^{\nu}\lambda_j + \frac{\rho^2 r}{128}\sum_{j=1}^{\nu}\lambda_j^2) \quad (13)
\end{aligned}$$

Recall that, for small arrays or high SNR, the pairwise error probability of DSTM is bounded by [3, 2]

$$P(G\to\hat{G}) \leq \left(\prod_{j=1}^{\nu}\lambda_j\right)^{-r}\left(\frac{\rho}{8}\right)^{-\nu r} \qquad (14)$$

To minimize this pairwise error probability bound in the high-SNR limit, we should therefore restrict attention to full diversity codes ($\nu = t$), in which case we can choose the *product distance* as the design criterion:

$$\Lambda_p(G,\hat{G}) = |(G-\hat{G})(G-\hat{G})^{\dagger}|^{1/t} \qquad (15)$$

where $|A|$ denotes the determinant of matrix A. Then (14) can be written as

$$P(G\to\hat{G}) \leq \left(\frac{\rho}{8}\Lambda_p(G,\hat{G})\right)^{-tr}\,. \qquad (16)$$

The pairwise error upper bounds (13) and (16) suggest that the design criteria for DSTM over quasi-static Rayleigh fading channel depend on the array size, which leads to the following code design criteria:

For small arrays, (16) shows that the rank and product distance are the dominant parameters in code performance, which is consistent with the rank and determinant criteria in [3, 2].

For large arrays, define $\Lambda_e = \|G-\hat{G}\|$ as the Euclidean distance of G and $\hat{G}$. (13) suggests that for $\rho d/16 \ll 1$, the pairwise error probability is dominated by Euclidean distance. In order to minimize the error probability for large arrays, we need to maximize the minimum Euclidean distance over all pairs of distinct matrices in $\mathcal{G}$. (13) also shows that codes that achieve good performance for large arrays may not have full diversity. A similar criterion to this Euclidean distance criterion is considered in [4], where the focus is on low-SNR performance rather than large arrays.

4 Code design for large arrays

We now use the Euclidean distance criterion to construct new differential unitary space-time codes appropriate for two transmit antennas and many receive antennas. Several codes with good performance for small arrays have already been presented in the literature, such as the differential scheme based on Alamouti's code in [7] and the group codes in [3]. In [8], we reconsidered the performance of these codes using the Euclidean distance criterion and then proposed some new codes for large arrays. For example, a new family of unitary codes referred to as Gram-Schmidt (GS) code is given by

$$G \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix} x & y \\ -zxy^{*} & z\end{bmatrix} \qquad (17)$$

with minimum Euclidean distance $\Lambda_e = 2|\sin(\pi/M)|$, where x, y and z are taken from the M-PSK constellation. For a given M-PSK constellation, the GS codes have the same minimum Euclidean distance as the Alamouti code [7], which has full diversity and takes the form (5), but the GS code achieves a rate that is 1.5 times larger. GS code is not a full diversity code.

In this section, we will propose some new codes that can achieve good performance for large arrays. Although under the Euclidean distance criterion, optimal codes may not necessarily have full diversity, here we will focus on full diversity codes in order to reduce the performance loss for small arrays.

Modified Gram-Schmidt (MGS) code

We first consider a class of unitary codes given by

$$G = \frac{1}{\sqrt{2}} \begin{bmatrix} x & x^p \\ x^q & -x^{p+q-1} \end{bmatrix} \tag{18}$$

where x is taken from M-PSK constellation. We call this code a Modified Gram-Schmidt (MGS) code for its structural similarity to the GS code. For simplicity, here we let p and q take integer values and choose them to maximize the resulting minimum Euclidean distance Λ_e. Generally, we can get optimal MGS codes that have better Euclidean distance than Alamouti code. For example, for rate $R = 3$ bps/Hz, the Alamouti code has $\Lambda_e = 0.7654$. By contrast, the full diversity MGS code with $p = 44$, $q = 34$ has $\Lambda_e = 1.3725$. At this rate, the GS code has $\Lambda_e = 1.4142$, which has slight gain relative to the MGS code, but the MGS code achieves full diversity.

Modified-Alamouti (MA) code

We now consider a family of codes which are closely related to the Alamouti code, given by

$$G = \frac{1}{\sqrt{2}} \begin{bmatrix} x & y \\ -y^* \triangle_{xy} & x^* \triangle_{xy} \end{bmatrix} \tag{19}$$

where x, y and $\triangle_{xy}$ are taken from M-PSK constellation, and $\triangle_{xy}$ is a function of x and y. We refer to this code as Modified-Alamouti (MA) code [8]. Here we let $\triangle_{xy} = x^m y^n$ where m and n are taken values from $[0 : 0.5 : M/2]$ and choose m and n to get the optimal full-diversity MA code with maximum Λ_e. For example, for $R = 3$ bps/Hz, the choice $m = 4$ and $n = 2.5$ yields a full diversity code with the largest Euclidean distance, $\Lambda_e = 1.2593$. MA codes often have better Euclidean distance than the Alamouti code, but typically have some performance loss relative to the Alamouti code for small arrays.

Table 1 summarizes the Euclidean distance performance of the codes discussed above. Also shown for comparison are the product distances of these codes. The parameters for MGS and MA codes are (p, q) and (m, n), respectively. We can conclude that codes that have large Λ_e may not necessarily have large Λ_p.

Table 1: Codes for Large Arrays ($t = 2$)

R	$\mathcal{G}$	Λ_e	Λ_p
1	Alamouti	2	2
	MGS(0,0)	2	2
	MA(0,0)	2	2
2	Alamouti	1.4142	1
	MGS(9,3)	2	1.4142
	MA(2,2)	2	1
3	Alamouti	0.7654	0.2929
	MGS(44,34)	1.3725	0.0341
	MA(4,2.5)	1.2593	0.0297
	GS	1.4142	0
4	Alamouti	0.3902	0.0761
	MGS(250,42)	0.9719	0.0315
	MA(8,6.5)	0.7654	0.0297

5 Numerical results

In this section, we provide the simulated bit-error rate (BER) performance of the aforementioned differential unitary codes for two transmit antennas and various numbers of receive antennas in a quasi-static Rayleigh flat fading channel.

In Fig. 1 and Fig. 2, we compare the BER performance of the MGS, MA, and Alamouti differential codes for rate $R = 2$ bps/Hz and rate $R = 3$ bps/Hz respectively. On the far right of the figures, we see that the Alamouti code outperforms the MGS and MA codes for $r = 1$. For $r = 2$, the performance of the three codes is considerably closer. For $r \geq 4$, however, the new codes perform better than the Alamouti code. In particular, at the BER of 10^{-3}, the MGS code outperforms the Alamouti code by approximately 1.3 dB for $r = 16, R = 2$, and by about 3 dB for $r = 16, R = 3$. These results provide additional evidence that Λ_e is an appropriate design criterion for large arrays, and that the new codes can improve the performance of DSTM in this regime. We further observe from the figures that Λ_e appears to be a good performance indicator even for relatively small arrays, such as $t = 2, r = 4$.

Fig. 3 compares the BER performance of the MGS, MA and GS differential codes for rate $R =$ 3bps/Hz. We can see that the MGS and MA codes have slight performance loss for large arrays but much performance gain for small arrays relative to the GS code.

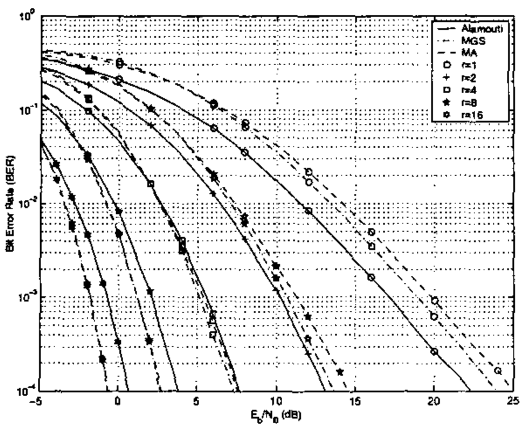

Figure 1: BER performance of Alamouti, MGS and MA codes for $R = 2$ bps/Hz and $t = 2$.

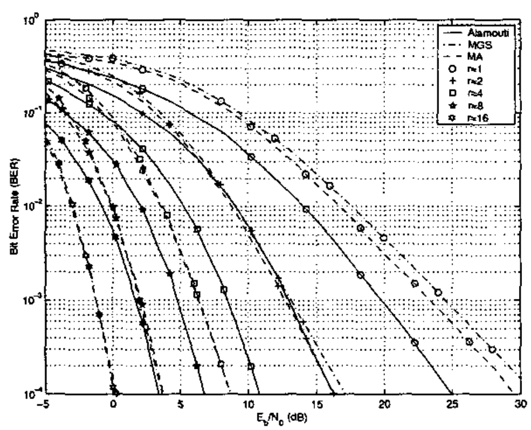

Figure 2: BER performance of Alamouti, MGS and MA codes for $R = 3$ bps/Hz and $t = 2$.

6 Conclusions

We have considered the design and analysis of differential space-time codes when the number of transmit or receive antennas is large. Based on the novel upper bound on the pairwise-error probability of DSTM in the presence of large numbers of antennas, we show that Euclidean distance is an appropriate code performance indicator in the large-array regime. For two transmit antennas and many receive antennas, we use the new design criterion to obtain some new differential codes with large minimum Euclidean distance. Simulations of bit-error-rate performance confirm that the new codes can achieve good performance for four or more receive antennas.

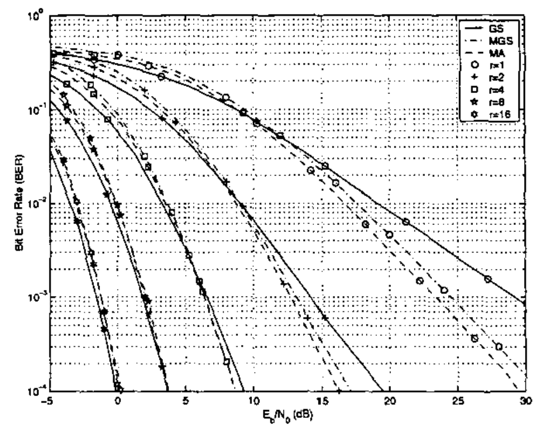

Figure 3: BER performance of GS, MGS and MA codes for $R = 3$ bps/Hz and $t = 2$.

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