Maximum likelihood single tone frequency estimation in a multipath channel

V.M. Baronkin, Y.V. Zakharov and T.C. Tozer

Abstract: The paper considers the maximum likelihood (ML) estimation of the frequency offset of a known signal received in a multipath channel. The ML algorithm and the Cramer–Rao lower bound are derived. An estimator possessing low computational load is proposed whose accuracy performance is close to that of the ML estimator over a wide frequency acquisition range.

1 Introduction

We consider the problem of estimating the frequency offset of a known signal received in a multipath channel. This has been addressed in recent publications [1, 2]. In [1] the broad problem has been considered where distinct paths introduce different frequency offsets to the signal received by an array of antennas. The proposed estimators, however, possessed a limited frequency acquisition range. In [2] an autocorrelation-based frequency estimator was proposed for diversity signals, assuming identical frequency offsets in different paths. Such a scenario corresponds to the case where oscillators in the transmitter and the receiver generate different reference frequencies with unknown frequency shift; at the same time, the frequency shift due to the Doppler effect is negligible. The estimator proposed in [2] also has a narrow frequency acquisition range.

We also consider the scenario where all the paths have the same frequency offset and the signal is received by one antenna. However, the problem is solved without any constraints upon the frequency offset. We derive the maximum likelihood (ML) frequency estimation algorithm and propose an approximation intended for real-time implementation. The proposed estimator is based on a coarse and fine search of the periodogram peak [3]. We also derive the Cramer–Rao lower bound (CRLB) for this case. Simulations show that the accuracy of the proposed algorithm is close to that of the ML frequency estimator for any signal-to-noise ratio (SNR) over a wide frequency acquisition range.

2 Signal model

Using complex-envelope notation, the observed signal can be modelled as

$$x(t) = A(t)e^{j\bar{\omega}_0 t} + n(t) \qquad t \in [0, T] \qquad (1)$$

where $\bar{\omega}_0$ is an unknown frequency offset and T is the signal duration. The additive complex white zero-mean Gaussian noise $n(t)$ has variance $\sigma^2 = E\{n(t)n^*(t)\}$, where $E\{\cdot\}$ denotes the statistical expectation and $(\cdot)^*$ denotes the complex conjugate. The complex envelope $A(t)$ can be represented as

$$A(t) = \int_{-\infty}^{\infty} h(u)s(t-u)du$$

where $s(t)$ is the transmitted signal and $h(t)$ is a channel impulse response

$$h(t) = \sum_{m=1}^{M} a_m \delta(t - \tau_m) \qquad (2)$$

where M is a number of paths, $\{\tau_m\}_{m=1}^{M}$ are delays of the paths, $\{a_m\}_{m=1}^{M}$ are complex-valued path amplitudes, and $\delta(t)$ is the Dirac delta function. Then the complex envelope $A(t)$ can be represented as

$$A(t) = \sum_{m=1}^{M} a_m \varphi_m(t) \qquad (3)$$

where $\varphi_m(t) = s(t - \tau_m)$.

We consider scenarios where the waveform $s(t)$ is known, i.e. a training sequence is transmitted for estimating the frequency offset. In some cases, the delays $\{\tau_m\}_{m=1}^{M}$ can be estimated before the frequency estimation [4, 5]. If the delays are known, eqn. 3 is a linear combination of known basis functions $\{\varphi_m(t)\}_{m=1}^{M}$. Then we have a set of linear parameters $\{a_m\}_{m=1}^{M}$ and a nonlinear parameter $\bar{\omega}_0$ to be estimated. Note that instead of the representation given in eqn. 2, a Fourier-series expansion of the channel impulse response $h(t)$ can also be used

$$h(t) = \sum_{m=1}^{M} a_m \psi_m(t) \qquad (4)$$

where $\{\psi_m(t)\}_{m=1}^{M}$ is a basis of linearly independent functions. Eqn. 4 is useful in channels with a time dispersion of discrete paths, for example, in underwater acoustic channels [6]. Then eqn. 3 holds if

$$\varphi_m(t) = \int_{-\infty}^{\infty} \psi_m(u)s(t-u)du$$

If, however, the delays are unknown we have a statistical problem with the time delays and frequency offset as unknown nonlinear parameters; the solution of this prob-

IEE Proceedings online no. 20010632

DOI: 10.1049/ip-com:20010632

Paper first received 15th March 2000 and in revised form 27th March 2001

V.M. Baronkin is with the N.N. Andreev Acoustics Institute, Moscow, Russia

Y.V. Zakharov and T.C. Tozer are with the Communications Research Group, Department of Electronics, University of York, York YO10 5DD, UK

lem is difficult to implement. To simplify it, we can exploit the canonical channel representation which states that the received signal can be represented arbitrarily accurately in terms of uniformly spaced multipath delays and Doppler shifts [6, 7]. In this case, the complex envelope is a linear combination

$$A(t) \approx \sum_{l=0}^{L} \sum_{k=-D}^{D} h_{k,l} u_{k,l}(t)$$

of a set of canonical basis waveforms

$$u_{k,l}(t) = s\left(t - \frac{l}{B}\right)\exp(j2\pi kt/T)$$

that are fixed *a priori* and do not depend on the actual physical delays and Doppler shifts; here B is the bandwidth of the waveform $s(t)$, $L \approx T_m B$, $D \approx TB_D$, T_m and B_D are the multipath and Doppler spread, respectively. Eqn. 3 holds if $M = (2D + 1)(L + 1)$, $m = (L + 1)k + l$, $a_m = h_{k,l}$ and $\varphi_m(t) = u_{k,l}(t)$. In particular, for a zero Doppler spread ($D = 0$) we have $\varphi_m(t) = s(t - m/B)$.

Thus, the model (eqn. 3) of the complex envelope is useful for many scenarios. Below we exploit this representation for the derivation of frequency estimates. We consider a discrete version of eqn. 1 obtained by using a sampling step $T_s = T/N$ where N is the number of samples observed. Then the signal model (eqn. 1) can be arranged in matrix form as

$$\boldsymbol{x} = \boldsymbol{W}(\omega_0)\boldsymbol{a} + \boldsymbol{n} \qquad \boldsymbol{W}(\omega_0) = \boldsymbol{\Lambda}(\omega_0)\boldsymbol{\Phi}$$

where $\boldsymbol{x}$ and $\boldsymbol{n}$ are $N \times 1$ column vectors with elements $x(iT_s)$ and $n(iT_s)$, respectively, $i = 0, ..., N - 1$, $\boldsymbol{a} = [a_1, ..., a_M]^T$ is an $M \times 1$ column vector, $\boldsymbol{\Phi}$ and $\boldsymbol{W}(\omega_0)$ are $N \times M$ matrices with elements $[\boldsymbol{\Phi}]_{i,m} = \varphi_{m,i}$ and $[\boldsymbol{W}(\omega_0)]_{i,m} = \varphi_{m,i}e^{j\omega_0 i}$, $\varphi_{m,i} = \varphi_m(iT_s)$, $\omega_0 = \bar{\omega}_0 T_s$, $[\cdot]^T$ denotes transposition and $\boldsymbol{\Lambda} = \text{diag}\{1, e^{j\omega_0}, ..., e^{j\omega_0(N-1)}\}$.

3 ML frequency estimator

To obtain the ML estimator we must maximise the probability density function of the observed data

$$p(\boldsymbol{x}|\boldsymbol{a},\omega) = \frac{1}{\pi^N \sigma^{2N}} \exp\left[-\frac{1}{\sigma^2}(\boldsymbol{x} - \boldsymbol{W}(\omega)\boldsymbol{a})^{\boldsymbol{H}}(\boldsymbol{x} - \boldsymbol{W}(\omega)\boldsymbol{a})\right]$$

where $(\cdot)^{\boldsymbol{H}}$ denotes complex transposition, or, equivalently, minimise the function $J(\boldsymbol{a}; \omega) = (\boldsymbol{x} - \boldsymbol{W}(\omega)\boldsymbol{a})^{\boldsymbol{H}}(\boldsymbol{x} - \boldsymbol{W}(\omega)\boldsymbol{a})$ over all possible $\boldsymbol{a}$ and ω. The matrix $\boldsymbol{W}(\omega)$ depends on ω and is independent of $\boldsymbol{a}$. Then we can write $\partial J(\boldsymbol{a}, \omega)/\partial \boldsymbol{a} = -[\boldsymbol{W}(\omega)^{\boldsymbol{H}}(\boldsymbol{x} - \boldsymbol{W}(\omega)\boldsymbol{a})]^*$ [8]. Setting $\partial J(\boldsymbol{a}, \omega)/\partial \boldsymbol{a}$ equal to zero we find the vector $\boldsymbol{a}$ minimising $J(\boldsymbol{a}; \omega)$ for a fixed frequency ω, this vector is

$$\hat{\boldsymbol{a}}(\omega) = \left(\boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{W}(\omega)\right)^{-1}\boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{x}$$

Substituting $\hat{\boldsymbol{a}}(\omega)$ back into $J(\boldsymbol{a}; \omega)$ we obtain

$$\min_{\boldsymbol{a}} J(\boldsymbol{a};\omega) = J(\hat{\boldsymbol{a}};\omega)$$
$$= \boldsymbol{x}^{\boldsymbol{H}}\boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{H}}\boldsymbol{W}(\omega)\left(\left(\boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{W}(\omega)\right)^{-1}\boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{x}\right.$$

To minimise $J(\hat{\boldsymbol{a}}; \omega)$ over ω we need to maximise the function

$$I_W(\omega) = \boldsymbol{x}^{\boldsymbol{H}}\boldsymbol{W}(\omega)\left(\boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{W}(\omega)\right)^{-1}\boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{x} \qquad (5)$$

which is a generalised periodogram. Note that $\boldsymbol{\Gamma} = \boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{W}(\omega) = \boldsymbol{\Phi}^{\boldsymbol{H}}\boldsymbol{\Phi}$ is the correlation matrix of the basis functions $\{\varphi_{m,i}\}_{m=1}^{M}$, its elements

$$\gamma_{mn} = \sum_{i=0}^{N-1} \varphi_{m,i}^* \varphi_{n,i} \qquad m, n = 1, ..., M \qquad (6)$$

are independent of the frequency ω. Note also, that $L_m(\omega) = [\boldsymbol{W}(\omega)^{\boldsymbol{H}}\boldsymbol{x}]_m$ is the output of a matched filter that correlates the received signal $x_i = x(iT_s)$ with a frequency shifted complex conjugate basis function $\varphi_{m,i}$

$$L_m(\omega) = \sum_{i=0}^{N-1} \varphi_{m,i}^* x_i e^{-j\omega i} \qquad (7)$$

If $\varphi_{m,i} = s(iT_s - \tau_m)$ then $L_m(\omega)$ is the output of a matched filter that correlates the received signal with a delayed and frequency shifted complex conjugate version of the known signal $s(t)$

$$L_m(\omega) = \sum_{i=0}^{N-1} s^*(iT_s - \tau_m) x_i e^{-j\omega i}$$

The ML estimator of the frequency ω_0 and amplitudes $\{a_m\}_{m=1}^{M}$ has the structure

$$\hat{\omega}_0 = \arg\max_{\omega}\{I_W(\omega)\} \qquad (8)$$

$$\hat{a}_m = \sum_{n=1}^{M} [\boldsymbol{\Gamma}^{-1}]_{mn} L_n(\hat{\omega}_0) \qquad m = 1, ..., M \qquad (9)$$

where by using eqn. 7 the generalised periodogram (eqn. 5) can be represented as

$$I_W(\omega) = \sum_{m=1}^{M} \sum_{n=1}^{M} [\boldsymbol{\Gamma}^{-1}]_{mn} L_m^*(\omega) L_n(\omega) \qquad (10)$$

If the correlation matrix $\boldsymbol{\Gamma}$ is diagonal, i.e. the basis functions $\{\varphi_{m,i}\}_{m=1}^{M}$ are orthogonal and $\gamma_{mn} \neq 0$ only if $m = n$, eqn. 10 is reduced to

$$I_W(\omega) = \sum_{m=1}^{M} \frac{1}{\gamma_{mm}} |L_m(\omega)|^2 \qquad (11)$$

and the amplitude estimate (eqn. 9) transforms to $\hat{a}_m = L_m(\hat{\omega}_0)/\gamma_{mm}$. Thus, the orthogonal basis functions allow simplification of the frequency estimation because in eqn. 11 we need to perform only M additions for every frequency ω, while eqn. 10 requires M^2 additions. For $M = 1$ the calculation of $I_W(\omega)$ is further simplified

$$I_W(\omega) = \frac{1}{\gamma_{11}} |L_1(\omega)|^2$$

For a one-path channel with unknown delay we can consider the following modification of the frequency estimation algorithm:

$$\hat{\omega}_0 = \arg\max_{\omega,\, m=1,...,M} \{|L_m(\omega)|^2\} \qquad (12)$$

The frequency estimate, eqn. 12, is defined by the position of the maximum of the path periodograms $|L_m(\omega)|^2$ for all possible frequencies and delays. Further, if $s(t)$ is a constant we obtain the classical periodogram whose maximiser is the ML estimate of a single tone frequency in white noise [3]

$$I_W(\omega) = \left|\sum_{i=0}^{N-1} x_i e^{-j\omega i}\right|^2$$

When implementing an approximation of the ML algorithm the matrix $\boldsymbol{\Gamma}^{-1}$ can be precomputed. Then for $M \ll N$ the complexity of the estimator is mainly due to calculation of $L_m(\omega)$. This calculation can be performed by using a DFT or FFT. The computational load increases with $K =$

$\Omega/\Delta\omega$, where Ω is the frequency acquisition range (the maximum range is $\Omega = 2\pi$) and $\Delta\omega$ is a frequency step used for calculation of $L_m(\omega_k)$, $\omega_k = k\Delta\omega$, $k = 0, ..., K-1$. To improve the frequency resolution, we need to decrease the frequency step $\Delta\omega$, i.e. to increase K. As a result, frequency estimators approximating eqn. 8 are considered to be too complicated for implementation even using an FFT for calculation (eqn. 7) and even for $M = 1$ [9]. For real-time implementation a simpler algorithm can be proposed.

4 Description of the proposed algorithm

We use an approach allowing us to reduce the number of frequencies for which the functions $L_m(\omega)$ are calculated. This approach is based on coarse and fine search of the periodogram peak [3]. The main purpose of the coarse search is to select a frequency interval where the maximised periodogram is a unimodal function. To this end, an FFT of a length P that is greater than N is usually used [10]. If an outlier occurs, the outcome of the coarse search will be any frequency in the frequency acquisition range Ω; this leads to a threshold effect at low SNRs, when the variance of the frequency error approaches $\Omega^2/12$ [3]. At SNRs higher than the SNR threshold the probability of an outlier is very small and the periodogram is usually a unimodal function over the selected frequency interval. Then, many optimisation techniques providing convergence when applied to unimodal functions can be exploited for the fine search, for example, Newton's method for finding the root of an equation [11] and different versions of three- and five-point interpolation [12–14]. However, the known fine search algorithms require nonlinear operations, which are difficult to implement. For the fine search we use an iterative DFT-based dichotomous procedure approaching the periodogram peak; this procedure was effectively applied to estimation of a frequency of a complex exponential signal in the AWGN channel [15]. The dichotomous algorithm does not require nonlinear operations and therefore is easier to implement. The frequency estimation algorithm can be described as follows.

For the coarse search, the data vector $\boldsymbol{x}$ of length N is extended to P samples by appending $(P - N)$ zeros. Then the periodogram samples are calculated using an FFT of length P over a grid of frequencies with a relatively large frequency step $\Delta\omega = 2\pi/P$, and the frequency ω_m that maximises the periodogram over this frequency set is identified

$$\omega_m = \arg\max_{\omega_k}\{I_W(\omega_k)\},\ \omega_k = k\Delta\omega,\ k = 0, ..., P-1$$

Simulations have shown that frequency errors of the ML and proposed estimators practically coincide when $P \geq 4N$. The value $\hat{\omega}_0 = \omega_m$ is a frequency estimate after the coarse search. The fine search locates a local maximum closest to ω_m. To this end, the peak sample in the periodogram and its two neighbours are selected: $I_1 = I_W(\omega_{m-1})$, $I_2 = I_W(\omega_m)$, $I_3 = I_W(\omega_{m+1})$. Then the algorithm performs the dichotomous search over a set of frequencies approaching the periodogram peak. This includes Q iterations consisting of

(i) $\Delta\omega = \Delta\omega/2$

(ii) if $I_3 > I_1$ then $I_1 = I_2$, $\hat{\omega}_0 = \hat{\omega}_0 + \Delta\omega$, else $I_3 = I_2$, $\hat{\omega}_0 = \hat{\omega}_0 - \Delta\omega$

(iii) a new sample of the periodogram on the frequency $\hat{\omega}_0$ is calculated by using a DFT: $I_2 = I_W(\hat{\omega}_0)$, go to (i)

After each iteration the frequency step $\Delta\omega$ decreases by a factor of 2. After all iterations, the final frequency estimate is $\hat{\omega}_0$, and the final frequency resolution is $\Delta\omega = 2\pi/(P2^Q)$. The number of frequencies for which the periodogram is calculated is $P + Q$ instead of $K = P2^Q$. This allows us substantially to reduce the computational load of the algorithm with respect to the direct use of the FFT of length K. The value of Q should be sufficiently large to obtain the final frequency resolution $\Delta\omega$ lower than the expected frequency error, for instance, lower than the CRLB.

5 Cramer–Rao lower bound

To derive the CRLB for variance of unbiased frequency estimates, consider the Fisher information matrix $\boldsymbol{I}(\boldsymbol{\Theta})$ with elements [8]

$$[\boldsymbol{I}(\boldsymbol{\Theta})]_{mn} = \frac{2}{\sigma^2}\mathrm{Re}\left\{\frac{\partial(\boldsymbol{W}(\omega)\boldsymbol{a})^{\boldsymbol{H}}}{\partial\Theta_m}\cdot\frac{\partial(\boldsymbol{W}(\omega)\boldsymbol{a})}{\partial\Theta_n}\right\} \tag{13}$$

where $\boldsymbol{\Theta} = [\mathrm{Re}\{\boldsymbol{a}\}\ \mathrm{Im}\{\boldsymbol{a}\}\ \omega]^T$ is a $(2M + 1) \times 1$ column vector of unknown parameters and $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ denote the real and imaginary parts, respectively, of a complex-valued number. Differentiating in eqn. 13, we obtain

$$\boldsymbol{I}(\boldsymbol{\Theta}) = \frac{2}{\sigma^2}\begin{bmatrix}\mathrm{Re}(\boldsymbol{\Gamma}) & -\mathrm{Im}(\boldsymbol{\Gamma}) & \mathrm{Re}(\boldsymbol{b})\\ \mathrm{Im}(\boldsymbol{\Gamma}) & \mathrm{Re}(\boldsymbol{\Gamma}) & \mathrm{Im}(\boldsymbol{b})\\ \mathrm{Re}(\boldsymbol{b}^T) & \mathrm{Im}(\boldsymbol{b}^T) & c\end{bmatrix} \tag{14}$$

where $\boldsymbol{\Gamma} = \boldsymbol{\Phi}^H\boldsymbol{\Phi}$ is the $M \times M$ correlation matrix with elements γ_{mn} from eqn. 6, the $M \times 1$ column vector $\boldsymbol{b} = j\boldsymbol{\Phi}^H\boldsymbol{K}\boldsymbol{\Phi}\boldsymbol{a}$ contains elements

$$[\boldsymbol{b}]_m = j\sum_{n=1}^{M} a_n \sum_{i=1}^{N-1} i\varphi_{m,i}^*\varphi_{n,i}$$

$$c = \boldsymbol{a}^{\boldsymbol{H}}\boldsymbol{\Phi}^{\boldsymbol{H}}\boldsymbol{K}^2\boldsymbol{\Phi}\boldsymbol{a} = \sum_{m=1}^{M}\sum_{n=1}^{M} a_m a_m^* \sum_{i=1}^{N-1} i^2\varphi_{m,i}^*\varphi_{n,i}$$

where $\boldsymbol{K}$ is an $N \times N$ diagonal matrix $\boldsymbol{K} = \mathrm{diag}\{0, ..., N-1\}$. The CRLB for frequency estimates $\hat{f}_0 = \hat{\omega}_0/2\pi$ is defined as

$$\sigma_f^2 = E\{(\hat{f}_0 - f_0)^2\}$$
$$\geq \sigma_{CR} = \frac{1}{4\pi^2}[\boldsymbol{I}^{-1}(\boldsymbol{\Theta})]_{2M+1,2M+1}$$

where $f_0 = \omega_0/2\pi$. To evaluate $[\boldsymbol{I}^{-1}(\boldsymbol{\Theta})]_{2M+1,2M+1}$ from eqn. 14, we use the partitioned matrix formula [8] which in this case gives

$$[\boldsymbol{I}^{-1}(\boldsymbol{\Theta})]_{2M+1,2M+1} = \frac{\sigma^2}{2(c - \boldsymbol{b}^{\boldsymbol{H}}\boldsymbol{\Gamma}^{-1}\boldsymbol{b})}$$

Note that the SNR is defined as $\eta = (\boldsymbol{a}^H\boldsymbol{\Gamma}\boldsymbol{a})/(N\sigma^2)$. Then, finally, we obtain

$$\sigma_{CR}^2 = \frac{\boldsymbol{a}^{\boldsymbol{H}}\boldsymbol{\Gamma}\boldsymbol{a}}{8\pi^2 N\eta(c - \boldsymbol{b}^{\boldsymbol{H}}\boldsymbol{\Gamma}^{-1}\boldsymbol{b})} \tag{15}$$

For the next analysis it is convenient to present eqn. 15 as

$$\sigma_{CR}^2 = \frac{1}{8\pi^2 N\eta}\cdot\frac{\boldsymbol{a}^{\boldsymbol{H}}\boldsymbol{\Gamma}\boldsymbol{a}}{\boldsymbol{a}^{\boldsymbol{H}}\boldsymbol{T}\boldsymbol{a}} \tag{16}$$

where the matrices $\boldsymbol{\Gamma} = \boldsymbol{\Phi}^H\boldsymbol{\Phi}$ and $\boldsymbol{T} = \boldsymbol{\Phi}^H\boldsymbol{K}[\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{\Phi}^H\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^H]\boldsymbol{K}\boldsymbol{\Phi}$ depend only on the basis functions (matrix $\boldsymbol{\Phi}$) and $\boldsymbol{I}$ is the identity matrix. For $M = 1$ it follows from eqn. 16 that the CRLB is

$$\sigma_{CR}^2 = \frac{1}{8\pi^2 N\eta d} \tag{17}$$

where $d = (\boldsymbol{s}^H\boldsymbol{K}^2\boldsymbol{s})/(\boldsymbol{s}^H\boldsymbol{s}) - (\boldsymbol{s}^H\boldsymbol{K}\boldsymbol{s})^2/(\boldsymbol{s}^H\boldsymbol{s})^2$ and $\boldsymbol{s} = [s(T_s), ..., s(NT_s)]^T$. The parameter d depends only on the transmitted signal. It can be used as a measure of the 'quality' of the signal; the higher d the lower is the CRLB in the AWGN channel. From eqn. 17 for $s(iT_s) = A > 0$ we obtain a well known bound for frequency estimates of a single tone in white noise [3]

$$\sigma_{CR}^2 = \frac{6}{\eta N(N^2-1)}$$

If $M > 1$ the analysis of the CRLB is more complicated because now the CRLB is a function of both the basis functions (matrix $\boldsymbol{\Phi}$) and the channel (vector $\boldsymbol{a}$). However, for a known transmitted signal, eqn. 16 allows us to calculate two bounds, $\sigma_{CR,min}^2$ and $\sigma_{CR,max}^2$, such that

$$\sigma_{CR,min}^2 \le \sigma_{CR}^2 \le \sigma_{CR,max}^2$$

These bounds correspond to 'best' and 'worst' channels, other channels will result in the CRLB between these bounds. From eqn. 16 we have

$$\sigma_{CR,min}^2 = \frac{1}{8\pi^2 N\eta\lambda_{max}} \quad \sigma_{CR,max}^2 = \frac{1}{8\pi^2 N\eta\lambda_{min}}$$

where λ_{min} and λ_{max} are, respectively, minimal and maximal eigenvalues of the matrix $\boldsymbol{T\Gamma}^{-1}$. Fig. 1 shows the bounds $\sigma_{CR,min}^2$ and $\sigma_{CR,max}^2$ as functions of M for sequences of maximal length (m-sequences) of lengths 15, 63 and 255. It can be seen that the bounds are tight, especially, for small M and long signals. Calculations of the bounds for other binary random sequences, in particular, with poorer autocorrelation properties, have shown similar results. As the accuracy performance of an ML estimator at high SNRs approaches the CRLB, results in Fig. 1 demonstrate that at high SNRs the performance of the ML frequency estimator has a weak dependence on the channel.

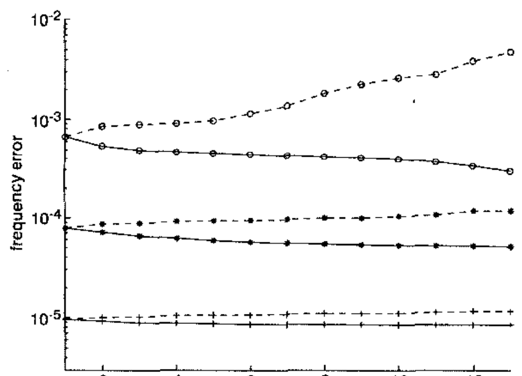

Fig. 1 *Minimal and maximal CRLB as a function of M*
—— minimal CRLB
- - - - maximal CRLB
—O— 15-symbol m-sequence
—*— 63-symbol m-sequence
—+— 255-symbol m-sequence

6 Simulation results

Fig. 2 shows the frequency error σ_f of the frequency estimator, defined by eqns. 8 and 10, as a function of SNR, averaged over 10000 simulation trials for $N = 18$, $M = 3$ and $f_0 = \omega_0/2\pi = 0.2$. The channel model has three paths with delays of $\tau_0 = 0$, $\tau_1 = T_s$, $\tau_2 = 2T_s$ and amplitudes $a_1 = \exp(-j0.4\pi)$, $a_2 = \exp(j0.1\pi)$ and $a_3 = \exp(j0.7\pi)$. The transmitted signal is a 15-symbol binary m-sequence; it has the autocorrelation $R(p) = \Sigma_{i=0}^{i=N-1} s^*(iT_s + pT_s)s(iT_s)$: $R(0) = 15$, $R(1) = 0$ and $R(2) = 1$. The frequency estimator is implemented by using (for the coarse search) an FFT of length $P = 64$ and the dichotomous fine search with $Q = 3$, $Q = 5$ or $Q = 7$ iterations. It can be seen that the increase of Q leads to improvement of the estimation accuracy at high SNRs; for $Q = 7$ the frequency error is close to the CRLB calculated by using eqn. 16. The SNR threshold (i.e. the SNR at which the frequency error departs from the CRLB) does not depend on the number of iterations Q used in the fine search; as was discussed in Section 4, the SNR threshold is defined by the coarse search. We have also performed a similar simulation for the frequency estimator defined by eqns. 8 and 10 and implemented by using an FFT of length 8192 without the dichotomous fine search; results of the simulation coincide with that presented in Fig. 2 for $Q = 7$. This is because the dichotomous algorithm uses the parameters $P = 64$ and $Q = 7$ providing the same frequency resolution as the direct FFT implementation of the ML estimator. It means that the proposed approximate ML estimator based on the dichotomous fine search allows significant simplification of implementation without losing the estimation accuracy.

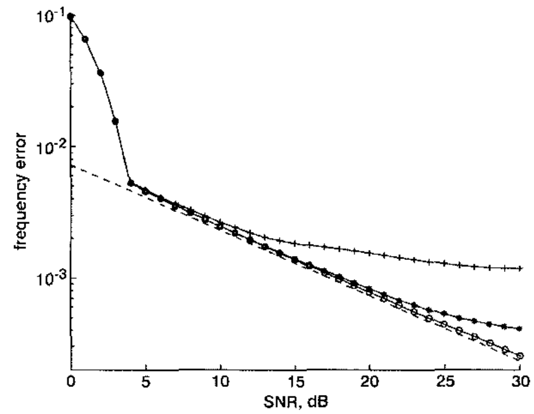

Fig. 2 *Dependence of the frequency error on SNR for the ML estimator with the dichotomous fine search with different numbers of iterations Q*
- - - - CRLB
—+— $Q = 3$
—*— $Q = 5$
—O— $Q = 7$

Fig. 3 compares the accuracy performance for two transmitted signals. Signal 1 is a 15-symbol m-sequence with autocorrelations $R(0) = 15$, $R(1) = 0$ and $R(2) = 1$, while signal 2 has autocorrelation with a higher level of sidelobes, $R(0) = 15$, $R(1) = 6$ and $R(2) = 7$. It can be seen that at high SNRs the performance of the estimator based on eqns. 8 and 10 for both signals is close to the CRLB, while at low SNRs we obtain an SNR threshold for signal 2 which is higher by about 1dB. For the transmitted signal with small sidelobe autocorrelations the frequency estimator based on eqns. 8 and 11 (not taking into account non-diagonal elements of the correlation matrix) gives a frequency error close to that of the estimator based on eqns. 8 and 10; a difference in the accuracy performance is seen only at high SNRs. However, for signal 2 which has relatively large autocorrelations, the accuracy performance is poor. The estimator based on eqn. 12, i.e. not taking into account multipath propagation, has an accuracy performance significantly poorer than that of the estimator based on eqns. 8 and 10.

Fig. 4 shows the frequency error as a function of the frequency f_0 being estimated for signal 1 and an SNR of 20dB. We see that the estimate is stable throughout the frequency acquisition range for all three estimators. Additional simulations for the scenarios considered have shown that the direct FFT-implementation and the dichotomous approximate implementation of the ML frequency estimators give close frequency errors.

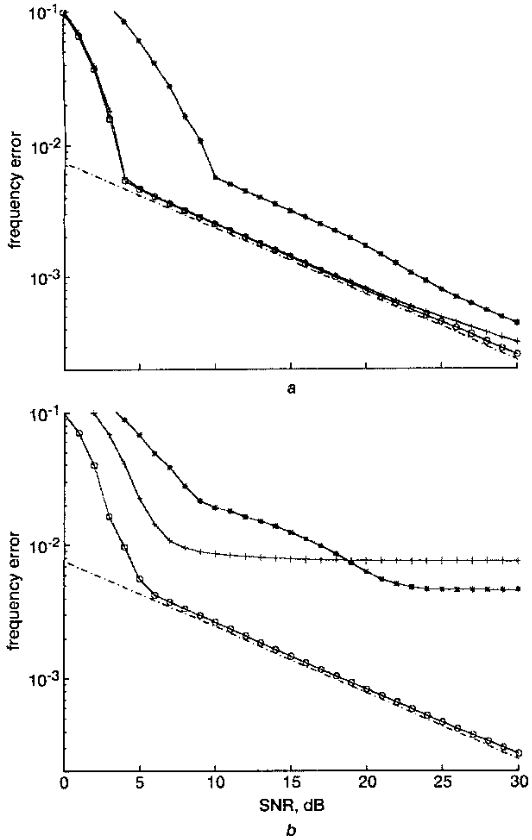

Fig. 3 *Dependence of the frequency error on SNR for three estimators with the dichotomous fine search*
---- CRLB
—O— frequency estimator based on eqns. 8 and 10
—+— frequency estimator based on eqns. 8 and 11
—✳— frequency estimator based on eqn. 12
a Signal 1
b Signal 2

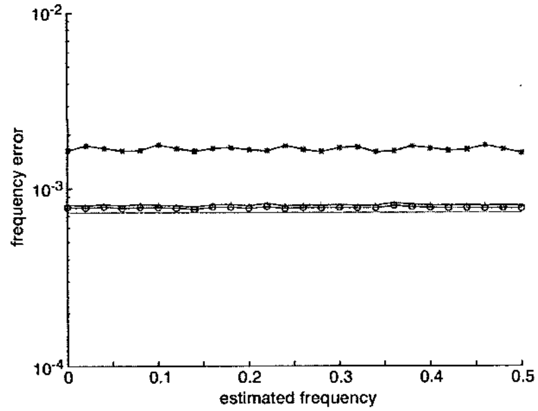

Fig. 4 *Dependence of the frequency error on the estimated frequency for three estimators with the dichotomous fine search*
SNR = 20dB
---- CRLB
—O— frequency estimator based on eqns. 8 and 10
—+— frequency estimator based on eqns. 8 and 11
—✳— frequency estimator based on eqn. 12

7 Conclusions

We have derived the ML algorithm for estimation of the frequency offset of a known signal received in a multipath channel together with the CRLB for this problem. We have shown that at high SNRs the performance of the ML frequency estimator has a weak dependence on the channel. We have proposed a computationally efficient frequency estimation algorithm based on the dichotomous fine search of the periodogram peak, whose accuracy practically coincides with that of the ML estimator over a wide range of SNR and frequencies.

8 Acknowledgment

The authors would like to thank the anonymous reviewers for their useful comments.

9 References

1 JAKOBSSON, A., SWINDLEHURST, A.L., and STOICA, P.: 'Subspase-based estimation of time delays and Doppler shifts', *IEEE Trans. Signal Process.*, 1998, **46**, (9), pp. 2472–2483

2 HEBLEY, M.G., and TAYLOR, D.P.: 'The effect of diversity on a burst-mode carrier-frequency estimator in the frequency-selective multipath channel', *IEEE Trans. Commun.*, 1998, **46**, (4), pp. 553–560

3 RIFE, D.C., and BOORSTYN, R.R.: 'Single-tone parameter estimation from discrete-time observations', *IEEE Trans. Inf. Theory*, 1974, **IT-20**, (5), pp. 591–598

4 GHOGHO, M., and SWAMI, A.: 'Estimation of frequency offset and Doppler rate in fading channels'. Proceedings of IEEE International Conference on *Personal wireless communications*, Jaipur, India, February 1999, pp. 105–109

5 ZAKHAROV, Y.V., ADLARD, J.F., and TOZER, T.C.: 'Detection of preamble part of random access burst in W-CDMA system'. Proceedings of 11th IEEE International Symposium on *Personal, indoor and mobile radio communications*, PIMRC'2000, London, UK, September 2000, Vol. 1, pp. 82–86

6 ZAKHAROV, Y.V., and KODANEV, V.P.: 'Multipath-Doppler diversity of OFDM signals in an underwater acoustic channel'. Proceedings of IEEE International Conference on *Acoustics, speech and signal processing*, ICASSP'2000, Istanbul, Turkey, June 2000, Vol. 5, pp. 2941–2944

7 SAYEED, A., and AAZHANG, B.: 'Joint multipath-Doppler diversity in mobile wireless communications', *IEEE Trans. Commun.*, 1999, **47**, (1), pp. 123–132

8 KAY, S.M.: 'Fundamentals of statistical signal processing: estimation theory' (Prentice Hall, Upper Saddle River, NJ, 1993)

9 YUAN, W.S., and GEORGHIADES, C.N.: 'Rapid carrier acquisition from baud-rate samples', *IEEE Trans. Commun.*, 1999, **47**, (4), pp. 631–641

10 PALMER, L.S.: 'Coarse frequency estimation using the discrete Fourier transform', *IEEE Trans. Inf. Theory*, 1974, **IT-20**, (1), pp. 104–109

11 ABATZOGLOU, T.J.: 'A fast maximum likelihood algorithm for frequency estimation of a sinusoid based on Newton's method', *IEEE Trans. Acoust. Speech Signal Process.*, 1985, **ASSP-33**, (1), pp. 77–89

12 MACLEOD, M.D.: 'Fast nearly ML estimation of the parameters of real or complex single tones or resolved multiple tones', *IEEE Trans. Signal Process.*, 1998, **46**, (1), pp. 141–148

13 QUINN, B.G.: 'Estimating frequency by interpolation using Fourier coefficients', *IEEE Trans. Signal Process.*, 1994, **42**, (5), pp. 1264–1268

14 QUINN, B.G.: 'Estimation of frequency, amplitude and phase from the DFT of a time series', *IEEE Trans. Signal Process.*, 1997, **45**, (3), pp. 814–817

15 ZAKHAROV, Y.V., and TOZER, T.C.: 'Frequency estimator with dichotomous search of periodogram peak', *Electron. Lett.*, 1999, **35**, (19), pp. 1608–1609